FROBENIUS HEIGHT OF PRISMATIC COHOMOLOGY WITH COEFFICIENTS

HAOYANG GUO AND SHIZHANG LI

Abstract. We study the behavior of Frobenius operators on smooth proper pushforwards of prismatic F-crystals. In particular we show that the i-th pushforward has its Frobenius height increased by at most i. Our proof crucially uses the notion of prismatic F-gauges introduced by Drinfeld and Bhatt–Lurie and its relative version, and we give a self-contained treatment without using the stacky formulation.

Contents

1. Introduction

Background and main result. Fix a prime number p, throughout this paper we let K be a p-adic discretely valued field with perfect residue field k, and let $\mathcal{O}_K$ be the ring of integers. For smooth proper p-adic formal schemes over $\mathcal{O}_K$, the prismatic cohomology introduced by Bhatt–Scholze in [BS22] (see also [BMS18] and [BMS19]) can be regarded as the universal cohomology theory. Recently there has been many works concerning "general coefficients" for this cohomology theory, known as *prismatic F-crystals*, to name a few: [MT20], [Wu21], [BS23], [DL21], [MW21], [DLMS22] and [GR22].

In this paper, we study the behavior of these general coefficients under smooth proper pushforwards. More precisely, we are interested in estimating the height of the induced Frobenius operator. Our main result is the following (combining Corollary 5.5 and Corollary 5.4):

Theorem 1.1 (Main Theorem)**.** *Let $f\colon X \to Y$ be a smooth proper morphism of smooth p-adic formal schemes over $\mathcal{O}_K$ of relative equidimension n, and let $(\mathcal{E}, \varphi_{\mathcal{E}})$ be an $\mathcal{I}$-torsionfree coherent prismatic F-crystal over X with Frobenius height in $[a, b]$.*

Then for each integer i, the i-th relative prismatic cohomology $R^i f_{\Delta,*}\mathcal{E}$ induces a coherent prismatic F-crystal over Y_{Δ}, whose $\mathcal{I}$-torsionfree quotient has Frobenius height in $[a+\max\{0,i-n\}, b+\min\{i,n\}]$.

Moreover there exists a Verschiebung operator

$$\psi_i \colon \mathcal{I}^{\otimes(i+b)} \otimes_{\mathcal{O}_{\Delta}} R^i f_{\Delta,*}\mathcal{E} \longrightarrow \varphi^*_{\Delta_Y} R^i f_{\Delta,*}\mathcal{E},$$

which is inverse to the Frobenius operator of $R^i f_{\Delta,*}\mathcal{E}$ up to multiplying $\mathcal{I}^{\otimes(i+b)}$.

According to [DLMS22, Theorem 1.3] or [GR22, Theorem A], any $\mathbb{Z}_p$-crystalline local system comes from an I-torsionfree coherent prismatic F-crystal. Moreover, in [GR22, §8] the first named author and Reinecke have described alternative approaches to showing that the induced Frobenius operator on $Rf_{\Delta,*}\mathcal{E}$ is an isogeny. The approach in this paper has the advantage that we can allow torsions in coefficients and cohomology, and we give precise bounds on Frobenius height.

Notice that prismatic F-crystals can give rise to usual crystalline F-crystals in characteristic p as well as local systems in characteristic 0. Our estimate of Frobenius height is compatible with the work of Kedlaya [Ked06, Thm. 6.7.1] bounding Frobenius slopes of rigid cohomology with coefficients as well as the work of Shimizu [Shi18, Theorem 5.10] controlling generalized Hodge–Tate weights of cohomology of local systems.

In the case of constant coefficient $\mathcal{E} = \mathcal{O}_{\Delta,X}$, this result has been established by Bhatt–Scholze in [BS22, §15]. In fact, our proof is in spirit a generalization of theirs: We found a canonical way to promote the F-crystal to a "Nygaard-filtered F-crystal", a.k.a. *prismatic F-gauge*, whose graded pieces are controlled and can be estimated by coherent cohomologies. The framework of prismatic F-gauges had been laid down by Drinfeld [Dri20] and Bhatt–Lurie [BL22a], [BL22b]. Many detailed discussions can be found in lecture notes by Bhatt [Bha22], and our paper is inspired by results therein. However we choose to not follow their stacky approach, and instead just discuss prismatic (F-)gauges and relative (F-)gauges using slightly more concrete terms.

In the remainder of this introduction, let us briefly indicate the proof idea, which will also allow us to summarize various sections of this article. In Section 2, we lay down some foundational discussions on homological algebras in graded or filtered setting. We in particular give several useful criteria of being "finite projecive" or "perfect" for graded/filtered complexes over graded/filtered rings, which might be of independent interest.

Constructions. As some preliminary discussions, in Section 3.1 we explain the existence of a standard t-structure on the category of prismatic (F-)crystals on X (see Definition 3.10), so that we have well-defined functors $R^i f_{\Delta,*}$ sending coherent crystals on X to those on Y, and expressions like "coherent" and "I-torsionfree" appearing in the main theorem above make sense.

Here comes the main construction. In Section 3.2, we give a definition of (*absolute*) (F-)*gauges* on the formal spectra of qrsp algebras S and show that the assignment $S \mapsto$ (F-)Gauge(Spf(S)) is a quasi-syntomic sheaf (see Proposition 3.30). By the unfolding process (c.f. [BMS19, Proposition. 4.31]), this gives rise to the notion of (F-)gauges on our X. Then we show the following:

Theorem 1.2 (= Theorem 3.32). *There is a fully faithful functor*

$$\Pi_X \colon \{I\text{-}\textit{torsionfree coherent } F\text{-}\textit{crystals on } X\} \longrightarrow \{\textit{coherent } F\text{-}\textit{gauges on } X\},$$

whose composition with the forgetful functor onto the underlying F-crystal is the identity.

The functor is defined by equipping an F-crystal with the "saturated Nygaardian filtration" on quasiregular semiperfectoid rings, and the essential work is to show that the filtration can be glued for quasi-syntomic topology[1]. This is inspired by the special case of $X = \mathrm{Spf}(\mathcal{O}_K)$ in [Bha22, Theorem 6.6.13] and extends the result in loc. cit. to arbitrary smooth p-adic formal schemes over $\mathrm{Spf}(\mathcal{O}_K)$ [2].

[1]The smoothness assumption is crucial in order to show that the saturated Nygaardian filtration glues. We do not know if this is true for F-crystals over general quasi-syntomic schemes, except when it is *admissible* of weight $[0,1]$ (cf. Theorem 3.54)

[2]Analogous to [Bha22, Proposition. 6.6.3], one can describe the essential image of the functor Π_X in terms of modules over the ring $A[u,t]/(ut-d)$, where $(A,(d))$ is a perfect prism.

Next in Section 4.2, we study the analogous notion of *relative (F-)gauges* on formal schemes Z that are quasi-syntomic over the reduction of some base prism (A, I) (see also [Tan22, §2]), which again satisfies the quasi-syntomic sheaf property (see Proposition 4.12). By a natural base change process, we have:

Theorem 1.3 (= Theorem 4.13)**.** *Let (A, I) be a bounded prism in Y_{Δ}. There is a natural functor*

$$BC\colon \{F\text{-gauges on } X\} \longrightarrow \{\text{relative } F\text{-gauges on } (X_{\overline{A}}/A)\}.$$

Moreover, we can associate a relative F-gauge with a *filtered Higgs field* and a *filtered flat connection.* This is analogous to Hodge–Tate comparison and de Rham comparison of prismatic cohomology in [BS22], except that we work with more general coefficients. Below for a quasi-syntomic formal scheme Z over $\overline{A}$, we let $\mathrm{FilHiggs}(Z/A)$ and $\mathrm{FilConn}(Z/\overline{A})$ be the categories of filtered Higgs fields over Z/A and filtered flat connections over $Z/\overline{A}$ separately. These are filtered derived enhancements of the usual notion of Higgs fields and flat connections (see Definition 4.18 and Definition 4.32 for details). Then we have the following constructions:

Theorem 1.4 (see §4.3 and §4.4 for details)**.** *Let (A, I) be a bounded prism, and let Z be a quasi-syntomic formal scheme over $\overline{A}$.*

(i) Hodge–Tate realization: *There is a natural functor*

$$\{\text{relative } F\text{-gauges on } (Z/A)\} \longrightarrow \mathrm{FilHiggs}(Z/A)$$

(ii) de Rham realization: *There is a natural functor*

$$\{\text{relative } F\text{-gauges on } (Z/A)\} \longrightarrow \mathrm{FilConn}(Z/\overline{A}).$$

Generalizing Nygaard–conjugate comparison in [BS22, Thm. 1.14.(2)], the functors above allow us to describe the graded pieces of a relative F-gauge in terms of the filtrations on the associated filtered Higgs complex and the filtered connection. For our application, we discuss the prismatic–Higgs relation in details and refer the reader to Proposition 4.26.

To summarize: at this point we have functorially attached filtrations to our prismatic F-crystal restricted to $(X_{\overline{A}}/A)$. It remains to study the induced filtration.

Properties. A key observation introduced in Section 3.3 is that there is a natural increasing *weight filtration* on the graded pieces of an absolute gauge E. An important feature for this weight filtration is that its graded pieces are generated by coherent complexes over $\mathcal{O}_X$. Precisely, we have:

Theorem 1.5 (= Theorem 3.45)**.** *Let X be a quasi-syntomic formal scheme, and let E be an F-gauge in perfect complexes over X. Then one can naturally associate the following data to E:*

- *two integers $a \leq b$,*
- *a perfect $\mathcal{O}_X$-complex $\mathrm{Red}_i(E)$ for each integer $i \in [a, b]$, and*
- *a finite exhaustive increasing filtration $\mathrm{Fil}_i^{\mathrm{wt}}$ on $\mathrm{Gr}^\bullet E$ of range $[a, b]$,*

such that the i-th graded piece $\mathrm{Gr}_i^{\mathrm{wt}}(\mathrm{Gr}^\bullet E)$ is naturally isomorphic to $\mathrm{Red}_i(E) \otimes_{\mathcal{O}_X} \mathrm{Gr}_N^\bullet \mathcal{O}_{\Delta}$.

The index of the weight filtration is called the *weights.* Moreover, when an F-gauge comes from an I-torsionfree coherent F-crystal with Frobenius height in $[a, b]$ as in Theorem 1.2, we can relate the weight filtrations on the graded piece $\mathrm{Gr}^\bullet E$ with the knowledge of the original Frobenius action; see Theorem 3.47. In particular, the weight of the associated F-gauge also lies in $[a, b]$. In fact, completely analogous statements hold as well in the relative setting and for filtered Higgs fields. Moreover the weight filtrations in absolute and relative settings are compatible under the realization functor in Theorem 1.4, see Proposition 4.17. We can thus understand cohomology of the filtered Higgs field M using coherent cohomology of $\mathrm{Red}_\bullet(M)$ and that of sheaves of differentials $\Omega^i_{Z/\overline{A}}$:

Theorem 1.6 (= Corollary 4.30)**.** *Assume Z is smooth of relative dimension n over $\overline{A}$, and let M be a filtered Higgs field of weight $[a, b]$. For each $i \in \mathbb{Z}$, the complex $R\Gamma(Z/A, \mathrm{Gr}_i M)$ admits a finite increasing exhaustive filtration of range $[a, b]$, such that each graded piece isomorphic to $R\Gamma(Z, \mathrm{Red}_u(M) \otimes_{\mathcal{O}_Z} \Omega^v_{Z/\overline{A}})$ for some $u, v \in \mathbb{Z}$.*

Combining these facts, we also get the following byproduct: the Higgs field coming from a coherent I-torsionfree F-crystal has nilpotent Higgs structure, and the order of nilpotence is bounded above by the length of the range of Frobenius heights.

On the other hand, recall in [BS22, §15] the authors showed that the graded pieces of Nygaard filtration of prismatic cohomology are conjugate filtrations of Hodge–Tate cohomology. Analogous results hold true in the setting of relative F-gauges: Namely for a relative F-gauge E over Z/A, its Frobenius structure induces isomorphisms between the graded pieces of $E^{(1)} := \varphi_A^* E$ and the filtrations of the associated filtered Higgs field (Proposition 4.26). Combine this with the analysis in Theorem 1.6, we can thus understand the cohomology of graded pieces of F-gauges using coherent cohomology.

To understand the filtration of F-gauges, another ingredient is the fact that we work in the world of perfect filtered/graded complexes for the entire construction process. In Section 4.4 we utilize this fact, coupled with the completeness of the relative Nygaard filtration on $\mathrm{R}\Gamma(X_{\overline{A}}/A, \mathbb{\Delta}^{(1)})$ and some descendability results, to show that the induced "Nygaard filtration" on the cohomology of $E^{(1)}$ is also complete. Thanks to the completeness, to bound the cohomology of the filtration of $E^{(1)}$, it suffices to do so on its graded pieces, and hence reduce the calculation to aforementioned Nygaard-conjugate isomorphism.

Finally in Section 5 we harvest the fruits of the above discussion and deduce several results, including those described in the above main theorem.

Byproducts and comments. At the end of this introduction, we mention several byproducts of our work and give some comments on related topics.

Remark 1.7 (Torsion of étale cohomology with coefficients)**.** Let X be a smooth p-adic formal scheme. As showed in [GR22, Theorem 6.1], taking derived direct image along proper smooth morphism commutes with étale realization functor on the category of prismatic F-crystal in perfect complexes (see also Theorem 5.8). Moreover, we observe (Lemma 3.16) that étale realization functor is t-exact. Combine the above observations with Theorem 1.1, we get the following refinement of "weak étale comparison" (c.f. [GR22, Thm. 6.1]):

Corollary 1.8 (= Theorem 5.8 + Corollary 5.9)**.** *Let $f\colon X \to Y$ be a smooth proper morphism between smooth formal schemes over $\mathrm{Spf}(\mathcal{O}_K)$, then derived pushforward of F-crystals in perfect complexes on X are F-crystals in perfect complexes on Y, and the following diagram commutes functorially:*

$$\begin{tikzcd} \text{F-Crys}^{\mathrm{perf}}(X) \arrow[r, "T(-)"] \arrow[d, "Rf_{\mathbb{\Delta},*}"'] & D^{(b)}_{lisse}(X_\eta, \mathbb{Z}_p) \arrow[d, "Rf_{\eta,*}"] \\ \text{F-Crys}^{\mathrm{perf}}(Y) \arrow[r, "T(-)"] & D^{(b)}_{lisse}(Y_\eta, \mathbb{Z}_p). \end{tikzcd}$$

Moreover we have $T(R^i f_{\mathbb{\Delta},*}\mathcal{E}) = R^i f_{\eta,*}(T(\mathcal{E}))$.

This allows us to study torsions of individual étale cohomology groups of a given lisse étale complex over $\mathbb{Z}_p$ that is *crystalline* [3], using the Frobenius structure and its bound on prismatic cohomology.

Remark 1.9 (p-divisible group and F-gauges)**.** Given a quasi-syntomic p-adic formal scheme X, it is shown in [ALB23, Thm 1.16] that there is a natural equivalence for the following categories:

$$\{\text{admissible prismatic } F\text{-crystals in vector bundles of height } [0,1] \text{ on } X\} \simeq \{p\text{-divisible groups on } X\}.$$

We show in Theorem 3.54 the aforementioned categories are further equivalent to the category of F-gauges in vector bundles of weight $[0, 1]$.

We also mention two natural variants of our results.

Remark 1.10 (Coherent F-gauge)**.** As shown in Lemma 3.35, an F-gauge that arises from an I-torsionfree coherent F-crystal satisfies some extra condition on the associated graded pieces, and we call such a condition *weakly reflexive*. This is related to the notion of *reflexiveness* introduced in [Bha22, Def. 6.6.4], but is slightly more genereal (as we do not assume the underlying F-crystal to be locally free). We expect that our method can be used to obtain an analogous result of Theorem 1.1 concerning these general F-gauges.

[3] We refer the reader to [Bha22, §1.2] for the notion and the related discussions.

Remark 1.11 (Positive characteristic case)**.** Although we only work with smooth p-adic formal schemes X over $\mathcal{O}_K$ and F-crystals/gauges on X, the entire article can be carried to the characteristic p setting: Namely we may instead assume X to be a smooth variety over a perfect field k, and consider cohomology of crystalline F-crystals/gauges on X. In this case, p-torsionfree coherent F-crystals naturally give rise to coherent F-gauges, and the Frobenius operator of their cohomology satisfies the same bound as in Theorem 1.1.

Notation and conventions. We work in infinity categorical language throughout the article, and assume the basics of prisms and quasi-syntomic rings, which we refer the reader to [Lur09], [Lur17], [BS22] and [BMS19] for details. Throughout the paper, the notation I is always a locally principal ideal related to prismatic constructions. To ease notation, as long as we believe there is no risk of confusion, we shall drop "$\widehat{-}$" in the notation of derived p-complete or (p, I)-complete version of the usual constructions: Theses include tensor products, pullback functors, Hodge and de Rham complexes.

A (decreasingly) filtered complex $\mathrm{Fil}^\bullet C$ is defined as an object in the filtered ∞-category $\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, D(\mathbb{Z}))$, where the latter is equivalent to the ∞-category of quasi-coherent complexes over the algebraic stack $\mathbb{A}^1/\mathbb{G}_m$ and is naturally equipped with a symmetric monoidal structure. A filtered algebra is then defined as an $\mathbb{E}_\infty$-algebra over the filtered ∞-category. Similarly, we can define the notion of a graded complex and a graded algebra in the derived content. Here for a filtered complex $\mathrm{Fil}^\bullet C$, we use $\mathrm{Gr}^\bullet C = \bigoplus \mathrm{Gr}^i C$ to denote its associated graded complex, and use $\mathrm{Fil}^\bullet C\langle i\rangle$ to denote its i-th shift of filtered degree, whose value at $n \in \mathbb{Z}$ is $\mathrm{Fil}^{i+n}C$. For our applications, we also use $\mathrm{Fil}_\bullet C$ and $\mathrm{Gr}_\bullet C$ to denote an increasingly filtered complex and its associated graded complex. For more details, we refer to [BMS19, §5.1] and [Bha22, §2.2.1] for brief introductions on filtered and graded objects.

We always fix K to be a complete p-adic discretely valued field with perfect residue field k, and let $\mathcal{O}_K$ be the ring of integers. To differentiate the notions, we use $\mathcal{E}$ to denote an (F-)crystal, and use E to denote an (F-)gauge.

Acknowledgements. The intellectual debt of this paper to Bhatt's notes [Bha22] should be obvious to readers. We have benefited from helpful conversations with the following mathematicians: Johannes Anschütz, Bhargav Bhatt, Heng Du, Hui Gao, Arthur-César Le Bras, Tong Liu, Shubhodip Mondal, Sasha Petrov, Emanuel Reinecke, Peter Scholze, Yupeng Wang, and Daxin Xu. We would like to express special thanks to Bhargav Bhatt for discussions on various parts of this paper. We are grateful to Tong Liu for mentioning this problem to us. During the writing of this paper, HG is supported by Max Planck Institut für Mathematik, and SL is supported by the National Key R & D Program of China No. 2023YFA1009701 and the National Natural Science Foundation of China (No. 12288201).

2. Graded and filtered homological algebra

In this section, we collect various preliminary results concerning graded (resp. filtered) complexes over graded (resp. filtered) rings. Throughout the whole section, we use the following notation:

Notation 2.1. Let A be a ordinary commutative ring with a finitely generated ideal $J \subset A$ such that A is derived J-complete.[4]

We shall refer to $\mathrm{CAlg}(\mathrm{Fun}(\mathbb{Z}^{\mathrm{disc}}, \mathcal{D}_{J\text{-comp}}(A)))$ as the category of derived J-complete graded $\mathbb{E}_\infty$-A-algebras indexed by $\mathbb{Z}$. We say such a graded algebra $B_\bullet = \oplus_{i\in\mathbb{Z}} B_i$ is *indexed by* $\mathbb{N}$ if $B_i = 0$ for all $i < 0$. We say such a graded algebra $B_\bullet = \oplus_{i\in\mathbb{Z}} B_i$ is *connective* if $B_i \in \mathcal{D}(A)$ is connective for all $i \in \mathbb{Z}$.

We shall refer to $\mathrm{CAlg}(\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{D}_{J\text{-comp}}(A)))$ as the category of derived J-complete (decreasing) filtered $\mathbb{E}_\infty$-A-algebras indexed by $\mathbb{Z}$. We say such a filtered algebra $\mathrm{Fil}^\bullet B$ is *indexed by* $\mathbb{N}$ if the induced map $\mathrm{Fil}^i B \to \mathrm{Fil}^{i-1} B$ is an equivalence for all $i \leq 0$. We say such a filtered algebra $\mathrm{Fil}^\bullet B$ is *connective* if $\mathrm{Fil}^i B \in \mathcal{D}(A)$ is connective for all $i \in \mathbb{Z}$. We use $D^{\mathrm{cn}}(A)$ to denote the category of connective complexes.

Whenever there is no risk of confusion, we shall ease the notation by dropping L and $\widehat{-}$ from the top of $\otimes$: Unless said otherwise, all the tensors are (J-complete) derived tensors.

[4]Later on, whenever we use the results of this section, we are always in the situation where either $A = \mathbb{Z}_p$ and $J = (p)$, or (A, I) is a bounded prism (whose choice shall be obvious from the context) and $J = (p, I)$.

2.1. **Finite projective modules and perfect complexes.** In this subsection, we shall define the notion of finite projective modules and perfect complexes in the context of graded or filtered algebras. We begin with discussing the notion of finite projective modules, for simplicity we shall restrict ourselves to the connective situation.

Digression 2.2 ([Lur17, §7.2.2])**.** Let R be a connective $\mathbb{E}_1$-ring, then define an object $M \in \mathcal{D}(R)$ to be a *finite projective module* if it is a retract of $R^{\oplus i}$ for some finite natural number i. This is equivalent to any of the following[5]:

- being compact projective in $\mathcal{D}^{\mathrm{cn}}(R)$;
- being flat over R and $\pi_0(M)$ is finitely generated over $\pi_0(R)$;
- the base change $\pi_0(R) \otimes_R M$ is finite projective over $\pi_0(R)$ (in the usual sense).

Suppose $R \to R'$ is a map of connective $\mathbb{E}_1$-rings inducing $\pi_0(R) \xrightarrow{\cong} \pi_0(R')$, then the base change functor induces an equivalence of homotopy categories $\mathrm{hProj}^{\mathrm{fin}}(R) \xrightarrow[\cong]{R' \otimes_R -} \mathrm{hProj}^{\mathrm{fin}}(R')$, see [Lur17, Corollary 7.2.2.19].

Let $R_\bullet$ be a connective graded $\mathbb{E}_1$-ring.

Definition 2.3. Define an object $M_\bullet \in \mathcal{DG}(R_\bullet)$ to be a graded finite projective module if it is a retract of a finite direct sum of $R_\bullet\langle i\rangle$'s. Here $(-)\langle i\rangle$ denotes shift of grading by i.

Recall that given a graded $\mathbb{E}_1$-ring $R_\bullet$, we may form its *direct sum totalization* (abbreviated as totalization below) $\bigoplus_{i\in\mathbb{Z}} R_i$, and similarly for a graded left $R_\bullet$-complex, we may form $\bigoplus_{i\in\mathbb{Z}} M_i$. These are $\mathbb{E}_1$-ring and left modules over it respectively.

Lemma 2.4. *An object $M_\bullet \in \mathcal{DG}(R_\bullet)$ is a graded finite projective module if and only if the totalization $\bigoplus_{i\in\mathbb{Z}} M_i$ is a finite projective module over $\bigoplus_{i\in\mathbb{Z}} R_i$.*

Proof. The "only if" part is clear, let us prove the "if" part. When the totalization is finite projective, choose a finite set of generators of $\pi_0(\bigoplus_{i\in\mathbb{Z}} M_i) = \bigoplus_{i\in\mathbb{Z}} \pi_0(M_i)$ and consider their grading-wise components, we see that there is a map $f \colon \bigoplus_{i=0}^m R_\bullet\langle n_i\rangle \to M_\bullet$ inducing surjection after passing to π_0 on each graded piece. Our condition implies that the totalization of the above map admits a section s. Using the following decomposition

$$\mathrm{Map}_{\bigoplus_{i\in\mathbb{Z}} R_i}(\bigoplus_{i\in\mathbb{Z}} M_i, \bigoplus_{i\in\mathbb{Z}} N_i) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Map}_{R_\bullet}(M, N_\bullet\langle i\rangle),$$

let us look at the component of s in degree 0, denote it by $s_0 \colon M_\bullet \to \bigoplus_{i=0}^m R_\bullet\langle n_i\rangle$. We need to show that $s_0 \circ f$ is an isomorphism on $M_\bullet$ or, what is the same, it induces an isomorphism on totalization. By construction, we know that it is an endomorphism of flat $(\bigoplus_{i\in\mathbb{Z}} R_i)$-module inducing an isomorphism on π_0, therefore it must induce isomorphisms on all π_i's by [Lur09, Definition 7.2.2.10]. □

Let $B_\bullet$ be a connective derived J-complete graded $\mathbb{E}_\infty$-A-algebras indexed by $\mathbb{Z}$.

Definition 2.5. Define an object $M_\bullet \in \mathcal{DG}_{J\text{-comp}}(B_\bullet)$ to be a *graded J-completely finite projective module* if its base change $A/J \otimes_A M_\bullet$ is a graded finite projective $A/J \otimes_A B_\bullet$-module.

Lemma 2.6. *The notion of being graded J-completely finite projective only depends on $V(J) \subset \mathrm{Spec}(A)$: Namely if J' is another finitely generated A-algebra with $V(J) = V(J')$, then being graded J-completely finite projective is equivalent to being graded J'-completely finite projective.*

Note that by [Sta23, Tag 091Q], the notion of being derived J-complete only depends on the underlying subset of $V(J)$.

Proof. It suffices to show that being graded J-completely finite projective implies being graded J^2-completely finite projective. Using Lemma 2.4 and the third characterization in Digression 2.2, it suffices to show: If $R \twoheadrightarrow R/J$ is a square-zero thickening of ordinary commutative rings with finitely generated kernel, then an R-complex is finite projective if and only if its base change to R/J is so. Let M be such an R-complex, using

[5]See [Lur17, Corollary 7.2.9 and Proposition 7.2.2.18] for the first two conditions, and the equivalence to the last condition follows from next sentence.

the triangle $J \otimes_{R/J} (R/J \otimes_R M) \to M \to R/J \otimes_R M$, we see that M is an ordinary R-module, and our statement follows from [Sta23, Tag 051C]. □

Remark 2.7. Let us point out that, even in the ungraded setting, being J-completely finite projective does not imply finite projective when the ring is not connective, see Remark 2.20 for such an example.

For our purpose, we need analogous results in the filtered setting. Via Rees's construction [Bha22, §2.2], we are led to consider analogs of the above in the graded setting. Let $\mathrm{Fil}^\bullet R$ be a connective filtered $\mathbb{E}_1$-ring.

Definition 2.8. Define an object $\mathrm{Fil}^\bullet M \in \mathcal{DF}(\mathrm{Fil}^\bullet R)$ to be a *filtered finite projective module* if it is a retract of a finite direct sum of $\mathrm{Fil}^\bullet R\langle i\rangle$'s. Here $(-)\langle i\rangle$ denotes shift of filtration by i.

Recall that given a filtered $\mathbb{E}_1$-ring $\mathrm{Fil}^\bullet R$, we may first view it as a graded ring by forgetting the arrows $\mathrm{Fil}^i \to \mathrm{Fil}^{i-1}$, then we can form its totalization $\bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i R$, this process is nothing but the Rees's construction. Similarly we may perform it for filtered modules over filtered rings.

Lemma 2.9. *An object $\mathrm{Fil}^\bullet M \in \mathcal{DF}(\mathrm{Fil}^\bullet R)$ is a filtered finite projective module if and only if its Rees's construction $\bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i M$ is a finite projective module over $\bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i R$.*

Proof. The proof is analogous to that of Lemma 2.4, since we still have the following decomposition

$$\mathrm{Map}_{\bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i R}\Big(\bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i M, \bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i N\Big) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Map}_{\mathrm{Fil}^\bullet R}(M, N_\bullet\langle i\rangle).$$

□

Let $\mathrm{Fil}^\bullet B$ be a connective derived J-complete filtered $\mathbb{E}_\infty$-A-algebras indexed by $\mathbb{Z}$.

Definition 2.10. Define an object $\mathrm{Fil}^\bullet M \in \mathcal{DF}_{J\text{-comp}}(\mathrm{Fil}^\bullet B)$ to be a filtered J-completely finite projective module if its base change $A/J \otimes_A M_\bullet$ is a filtered finite projective $A/J \otimes_A \mathrm{Fil}^\bullet B$-module.

Following the exact same proof of Lemma 2.6, we get the following:

Lemma 2.11. *The notion of being filtered J-completely finite projective only depends on $V(J) \subset \mathrm{Spec}(A)$: Namely if J' is another finitely generated A-algebra with $V(J) = V(J')$, then being filtered J-completely finite projective is equivalent to being filtered J'-completely finite projective.*

Next, let us define perfect complexes in the context of graded or filtered algebras.

Digression 2.12 ([Lur17, §7.2.4])**.** We begin by recalling [Lur17, Definition 7.2.4.1]: Let R be an $\mathbb{E}_1$-ring, then the subcategory of left perfect R-complexes in the ∞-category of left R-complexes is defined as the smallest stable subcategory containing R and closed under retracts, objects of this subcategory are called left perfect R-complexes. Immediately after this definition, Lurie proved [Lur17, Proposition 7.2.4.2]: In the above setting, left perfect R-complexes are the same as compact objects among left R-complexes, and they generate the ∞-category of left R-complexes. In fact there is a third characterization of perfectness [Lur17, Proposition 7.2.4.4]: it is equivalent to being dualizable.

Mimicking the above, we arrive at the following:

Definition 2.13. Let $R_\bullet$ be a graded $\mathbb{E}_1$-ring, then the subcategory of graded left perfect $R_\bullet$-complexes in the ∞-category of graded left $R_\bullet$-complexes is defined as the smallest stable subcategory containing $R_\bullet\langle i\rangle$ for all i and closed under retracts. We denote this subcategory by $\mathcal{G}\mathrm{LMod}_{R_\bullet}^{\mathrm{perf}}$, objects of which are called graded left perfect $R_\bullet$-complexes.

Here is an analog of [Lur17, Proposition 7.2.4.2].

Proposition 2.14. *Let $R_\bullet$ be a graded $\mathbb{E}_1$-ring. Then:*

1. *The ∞-category $\mathcal{G}\mathrm{LMod}_{R^\bullet}$ is compactly generated.*
2. *An object of $\mathcal{G}\mathrm{LMod}_{R^\bullet}$ is compact if and only if it is perfect.*

Moreover the property of being perfect/compact is equivalent to the totalization being perfect/compact as a complex over the totalization of $R_\bullet$.

Proof. The proof of (1) and (2) are exactly the same as that in loc. cit: Tracing through the proof there, we just need to observe that if $\mathrm{Map}_{R_\bullet}(R_\bullet\langle i\rangle[j], M_\bullet) = 0$ for all i, j, then $M_\bullet = 0$. As for the last statement: it simply follows from the fact that taking totalization has a right adjoint preserving filtered colimit:

$$\mathrm{LMod}_{\bigoplus_i R_i} \to \mathcal{G}\mathrm{LMod}_{R_\bullet}, M \mapsto M \otimes_{(\bigoplus_i R_i)} \left(\bigoplus_{j\in\mathbb{Z}} R_\bullet\langle j\rangle\right),$$

where the map $\bigoplus_i R_i \to \bigoplus_{j\in\mathbb{Z}} R_\bullet\langle j\rangle$ is the canonical map realizing the source as the deg $= 0$ piece of the target. $\square$

Remark 2.15. If $R_\bullet$ is a graded discrete commutative ring, then graded $R_\bullet$-complexes are the same as quasi-coherent complexes on the stack $[\mathrm{Spec}(\bigoplus_i R_i)/\mathbb{G}_m]$, whose perfectness is defined by that of its pullback to $\mathrm{Spec}(\bigoplus_i R_i)$. This is a well-defined notion due to flat descent of perfectness (see [Sta23, Tag 066X] and [Sta23, Tag 068T]). Our proposition above shows that in this special case, the perfectness in classical sense agrees with our definition here.

Via Rees's construction [Bha22, §2.2], we can transport the above discussion from graded setting to filtered setting.

Definition 2.16. Let $\mathrm{Fil}^\bullet S$ be a filtered $\mathbb{E}_1$-ring, then we may view it as a graded $\mathbb{E}_1$-ring. We then define the ∞-category of filtered left $\mathrm{Fil}^\bullet S$-modules as $\mathcal{F}\mathrm{LMod}_{\mathrm{Fil}^\bullet S} := \mathcal{G}\mathrm{LMod}_{\mathrm{Fil}^\bullet S}$ (c.f. [Bha22, Proposition 2.2.6]). We then define a filtered left complex $\mathrm{Fil}^\bullet M$ to be perfect if its image under the above equivalence is perfect. Since the above is an equivalence, we immediately see that the subcategory of filtered left perfect $\mathrm{Fil}^\bullet S$-complexes can be alternatively defined as the smallest stable subcategory containing $(\mathrm{Fil}^\bullet S)\langle i\rangle$ for all i and closed under retracts. Here once again, we use $(-)\langle i\rangle$ to denote shift of filtration by i.

The following is a simple translation of Proposition 2.14 via the Rees equivalence.

Proposition 2.17. *Let $\mathrm{Fil}^\bullet S$ be a filtered $\mathbb{E}_1$-ring. Then:*

(1) The ∞-category $\mathcal{F}\mathrm{LMod}_{\mathrm{Fil}^\bullet S}$ is compactly generated.
(2) An object of $\mathcal{F}\mathrm{LMod}_{\mathrm{Fil}^\bullet S}$ is compact if and only if it is perfect.

Moreover the property of being perfect/compact is equivalent to its underlying Rees construction being perfect/compact as a $\mathrm{Rees}(\mathrm{Fil}^\bullet S)$-complex.

We also get the following useful property of filtered left perfect complexes over a complete filtered ring.

Proposition 2.18. *Let $\mathrm{Fil}^\bullet S$ be a filtered $\mathbb{E}_1$-ring which is complete with respect to its filtration. Any filtered left perfect $\mathrm{Fil}^\bullet S$-complex is complete with respect to its filtration.*

Proof. We simply observe that the subcategory spanned by filtered complexes being complete with respect to its filtration is a stable subcategory containing $(\mathrm{Fil}^\bullet S)\langle i\rangle$ for all i and closed under retracts. Therefore it must contain the subcategory of perfect complexes. $\square$

Since we work with derived complete complexes, we need to introduce the following variant. Let $\mathrm{Fil}^\bullet B$ be a derived J-complete (decreasing) filtered $\mathbb{E}_\infty$-A-algebras indexed by $\mathbb{Z}$.

Definition 2.19. Define an object $\mathrm{Fil}^\bullet M \in \mathcal{DF}_{J\text{-comp}}(\mathrm{Fil}^\bullet B)$ to be *filtered J-completely perfect* if its base change $A/J \otimes_A \mathrm{Fil}^\bullet M$ is filtered perfect over $A/J \otimes_A \mathrm{Fil}^\bullet B$. We denote this full subcategory by $\mathcal{DF}^{\mathrm{perf}}_{J\text{-comp}}(\mathrm{Fil}^\bullet B)$.

Note that the inclusion $\mathcal{DF}^{\mathrm{perf}}(\mathrm{Fil}^\bullet B) \subset \mathcal{DF}^{\mathrm{perf}}_{J\text{-comp}}(\mathrm{Fil}^\bullet B)$ need not be an equivalence, below we sketch an example due to Bhatt and Reinecke which we learn from private communications.

Remark 2.20. Bhargav Bhatt and Emanuel Reinecke gave the following example where perfectness and p-completely perfect genuinely differs. Let us sketch their idea: Let $R = \widehat{\mathrm{dR}}_{\mathbb{Z}_p\langle x\rangle/\mathbb{Z}_p}$, and take any connection $\nabla\colon \mathbb{Z}_p\langle x\rangle \to \mathbb{Z}_p\langle x\rangle \cdot dx$ such that

(1) $\nabla \equiv d$ modulo p; and

(2) $\nabla = 0$ has no solution.

Then we claim that the complex $M := [\mathbb{Z}_p\langle x\rangle \xrightarrow{\nabla} \mathbb{Z}_p\langle x\rangle \cdot dx]$ gives an example of a p-completely perfect R-complex which is not perfect. It is p-completely free of rank 1 by the condition (1) above, let us show it is not perfect. Suppose otherwise, consider $M \otimes^L_R \mathbb{Z}_p\langle x\rangle$, which is a perfect $\mathbb{Z}_p\langle x\rangle$-complex, whose reduction modulo p is $\mathbb{F}_p[x]$ due to condition (1) above. Therefore we see that the $\mathbb{Z}_p$-complex $M \otimes^L_R \mathbb{Z}_p\langle x\rangle$ is non-canonically isomorphic to $\mathbb{Z}_p\langle x\rangle$. On the other hand, since $\mathrm{H}^0(R[1/p]) = \mathbb{Q}_p$, one can show that the $R[1/p]$-complex $\mathbb{Q}_p\langle x\rangle$ admits an exhaustive increasing filtration with graded pieces given by (infinite) direct sums of $R[1/p]$. Therefore we see that $\big(M \otimes^L_R \mathbb{Z}_p\langle x\rangle\big)[1/p]$ admits an exhaustive increasing filtration with graded pieces given by direct sums of $M[1/p]$, which has only cohomology in degree 1 by condition (2) above. This gives a contradiction as desired. In fact, one can equip both M and R with Hodge filtration so that it becomes an example illustrating $\mathcal{DF}^{\mathrm{perf}}(\mathrm{Fil}^\bullet B) \subsetneq \mathcal{DF}^{\mathrm{perf}}_{J\text{-comp}}(\mathrm{Fil}^\bullet B)$, with $(A, J) = (\mathbb{Z}_p, (p))$.

In familiar situations, the difference between perfectness and complete perfectness disappears.

Lemma 2.21 ([Sta23, Tag 09AW])**.** *Let notation be as in Definition 2.19. If R is concentrated in degree 0 and is J-adically complete, then a J-complete R-complex is perfect if and only if it is J-completely perfect.*

It is easy to see that in the above situation, the difference between being finite projective versus being J-completely finite projective also disappears. Let us show that the same is true as long as the R is a connective derived J-complete $\mathbb{E}_\infty$-A-algebra.

Proposition 2.22. *Let R be a connective derived J-complete $\mathbb{E}_\infty$-A-algebra. Let $M \in \mathcal{D}_{J\text{-}comp}(R)$, then*

(1) it is finite projective if and only if it is J-completely finite projective;
(2) it is perfect if and only if it is J-completely perfect.

Proof. The only if part is clear. Let us prove the if part for (1). Our assumption implies that M is connective, and $\pi_0(M)/J$ is finite projective over $\pi_0(R)/J$. Choose a finite set of generators and lift to $\pi_0(M)$, we obtain a map $f\colon R^{\oplus n} \to M$. Our task is to show that this map admits a splitting given that it does so after applying $A/J \otimes_A -$. Choose a finite set of generators $J = (a_1, \dots, a_m)$, for each k, let us consider the base change of f along $\mathrm{Kos}(R; a_i^k) \otimes_R -$. Combining [Sta23, Tag 051C] and [Lur17, Corollary 7.2.2.19], we see that the map $\mathrm{Kos}(R; a_i^k) \otimes_R f$ is again a map from finite free $\mathrm{Kos}(R; a_i^k)$-complex to a finite projective $\mathrm{Kos}(R; a_i^k)$-complex inducing surjection on π_0, therefore it admits a section. Now we compute the mapping space:

$$\mathrm{Map}_R(M, R^{\oplus n}) \cong \lim_k \mathrm{Map}_R(M, \mathrm{Kos}(R; a_i^k)^{\oplus n}) \cong \lim_k \mathrm{Map}_{\mathrm{Kos}(R;a_i^k)}(\mathrm{Kos}(M; a_i^k), \mathrm{Kos}(R; a_i^k)^{\oplus n}),$$

and observe that the last term has π_0, for each k, given by $\mathrm{Hom}_{\pi_0(R)/(a_i^k)}(\pi_0(M)/(a_i^k), \pi_0(R)/(a_i^k)^{\oplus n})$.[6] In particular, by Milnor's exact sequence $\mathrm{Hom}_R(M, R^{\oplus n}) \twoheadrightarrow \lim_k \mathrm{Hom}_{\pi_0(R)/(a_i^k)}(M/(a_i^k), \pi_0(R)/(a_i^k)^{\oplus n})$, we may successively lift the section when $k = 1$ along the inverse system. In conclusion, we have found a map $s\colon M \to R^{\oplus n}$ such that $\pi_0(R/J) \otimes_R (f \circ s)$ is identity. This implies first that $A/J \otimes_A (f \circ s)$, being an endomorphism of a finite projective $A/J \otimes_A R$-module inducing identity on π_0, is an isomorphism. Then using derived Nakayama, this shows that $f \circ s$ is actually an isomorphism.

It is straightforward to deduce (2) out of (1): Let us perform induction on the length of the $\pi_0(R)/J$-Tor-amplitude interval of $\pi_0(R)/J \otimes_R M$. When the length is 0, it follows from (2) that M is a shift of a finite projective R-module. Suppose the case of length ℓ is proved, then without loss of generality assume the said Tor-amplitude is in $[-\ell - 1, 0]$. By derived completeness, we see that M is connective. Let us choose a map $R^{\oplus n} \to M$ inducing a surjection on $\pi_0(-)/J$. Then the cone is derived J-complete, and the said Tor-amplitude length is drop by 1, so we win by induction. □

Later on in this section, we shall see that the difference between graded/filtered perfectness (resp. finite projective) and completely perfectness (resp. completely finite projective) also disappears under mild condition: see Corollary 2.26, Corollary 2.27, Proposition 2.30, and Proposition 2.35.

[6] This follows from the fact that $\mathrm{Kos}(M; a_i^k)$ is a finite projective $\mathrm{Kos}(R; a_i^k)$-complex.

2.2. **Graded homological algebra.** Throughout this subsection, let $B_\bullet$ be a derived J-complete graded $\mathbb{E}_\infty$-A-algebras *indexed by* $\mathbb{N}$. To start, we first observe the following reduction functor:

Construction 2.23. Since $B_\bullet$ is indexed by $\mathbb{N}$, projecting to its degree 0 piece is a map of derived J-complete graded $\mathbb{E}_\infty$-A-algebras. Base changing (J-completely) along this map gives us a "reduction functor":

$$\mathrm{Red}_B := \mathcal{DG}_{J\text{-comp}}(B_\bullet) \xrightarrow{-\otimes_{B_\bullet} B_0} \mathcal{DG}_{J\text{-comp}}(B_0);$$

$$M^\bullet \longmapsto M^\bullet \otimes_{B_\bullet} B_0.$$

Here let us emphasize again that $\otimes$ denotes the J-completely derived tensor product. The i-th degree piece of $\mathrm{Red}_B(-)$ is denoted as $\mathrm{Red}_{i,B}(-)$. When the $B_\bullet$ is clear from the context, we omit B in the subscripts and use Red and Red_i to abbreviate Red_B and $\mathrm{Red}_{i,B}$.

We are interested in the following subcategory of graded $B_\bullet$-complexes.

Definition 2.24. We say an $M_\bullet \in \mathcal{DG}_{J\text{-comp}}(B_\bullet)$ has *bounded below grading* if $M_i = 0$ for all $i \ll 0$.

Using the reduction functor, we now introduce a *canonical weight* filtration on those $M_\bullet$'s with bounded below grading.

Proposition 2.25. *Let $M_\bullet \in \mathcal{DG}_{J\text{-comp}}(B_\bullet)$ has bounded below grading, there is a unique increasing filtration $\mathrm{Fil}_i^{\mathrm{wt}}(M_\bullet)$ on $M_\bullet$ satisfying the following axioms:*

(1) The filtration is exhaustive;

(2) the filtration "starts at 0": i.e. $\mathrm{Fil}_{\ll 0}^{\mathrm{wt}}(M_\bullet) = 0$; and

(3) the graded piece $\mathrm{Gr}_i^{\mathrm{wt}}(M_\bullet) \simeq N_i \otimes_{B_0} B_\bullet$ where N_i is a J-complete B_0-complex with grading i.

Moreover, the induced filtration $\mathrm{Fil}_i^{\mathrm{wt}}(\mathrm{Red}(M_\bullet)) := \mathrm{Red}(\mathrm{Fil}_i^{\mathrm{wt}}(M_\bullet))$ is the one induced by grading:

$$\mathrm{Fil}_i^{\mathrm{wt}}(\mathrm{Red}(M_\bullet)) \cong \bigoplus_{j \le i} \mathrm{Red}_j(M_\bullet).$$

In particular, there are natural isomorphisms $N_i \cong \mathrm{Red}(\mathrm{Gr}_i^{\mathrm{wt}}(M_\bullet)) \cong \mathrm{Gr}_i^{\mathrm{wt}}(\mathrm{Red}(M_\bullet)) \cong \mathrm{Red}_i(M_\bullet)$.

Proof. We show the uniqueness first. Let c be the least integer such that $M_c \neq 0$, which exists by assumption. Let $\mathrm{Fil}_*(M_\bullet)$ be any filtration satisfying the list of axioms. Let d be the least integer such that $\mathrm{Fil}_d(M_\bullet) \neq 0$, which exists as $M_\bullet \neq 0$ and the exhaustive axiom (1). By our condition (3), we see that the grading $\le d$ piece of the map $\mathrm{Fil}_i(M_\bullet) \to \mathrm{Fil}_{i+1}(M_\bullet)$ is an isomorphism whenever $i \ge d$. Our assumption on d implies that

$$0 \neq N_d \cong (\mathrm{Gr}_d(M_\bullet))_d \cong (\mathrm{Fil}_{\ge d}(M_\bullet))_d \cong M_d$$

and

$$0 = (\mathrm{Fil}_{\ge d}(M_\bullet))_{<d} \cong M_{<d}.$$

where the last isomorphisms follow from the axiom (1). This shows that $d = c$. By the above reasoning, we necessarily have $\mathrm{Fil}_{<c}(M_\bullet) = 0$ and $\mathrm{Fil}_c(M_\bullet) = M_c \otimes_{B_0} B_\bullet$ with map to $M_\bullet$ induced by the natural map $M_c = M_c \subset M_\bullet$ of graded B_0-complexes. We observe that the choice of $\mathrm{Fil}_c(M_\bullet)$ is exactly so that $(M_\bullet/\mathrm{Fil}_c(M_\bullet))_{\le c} = 0$.

Suppose $\mathrm{Fil}_{\le d}(M_\bullet)$ is determined such that $(M_\bullet/\mathrm{Fil}_d(M_\bullet))_{\le d} = 0$, then to determine Fil_{d+1} is the same as to determine $\mathrm{Fil}_{d+1}(M_\bullet/\mathrm{Fil}_d(M_\bullet))$, which we have seen that it must be given by $(M_\bullet/\mathrm{Fil}_d(M_\bullet))_{d+1} \otimes_{B_0} B_\bullet$. Moreover this unique choice exactly makes that $(M_\bullet/\mathrm{Fil}_{d+1}(M_\bullet))_{\le(d+1)} = 0$. Therefore we have the algorithm of determining $\mathrm{Fil}_*(M_\bullet)$, which is uniqueness.

To see existence, we simply notice that the filtration produced by the algorithm from the previous paragraph satisfies axioms by how it is constructed: (2) and (3) is easy to see, the exhaustive axiom (1) follows from the property that $(M_\bullet/\mathrm{Fil}_d(M_\bullet))_{\le d} = 0$.

The induced filtration on $\mathrm{Red}(M_\bullet)$ is an increasing exhaustive filtration, starting at 0, whose graded pieces are concentrated in different grading. There is no choice but the filtration induced by grading. $\square$

Corollary 2.26. *Suppose that $B_\bullet$ is indexed by $\mathbb{N}$. Let $M_\bullet \in \mathcal{DG}_{J\text{-}comp}(B_\bullet)$, then it is graded ($J$-completely) perfect if and only if it is has bounded below grading and $\mathrm{Red}_B(M_\bullet) \in \mathcal{DG}_{J\text{-}comp}(B_0)$ is graded (J-completely) perfect.*

Proof. Only if part follows from the fact that $B_\bullet$ is indexed by $\mathbb{N}$ (hence has bounded below grading) and that Red_B is defined as a base change hence preserves properties of being graded (J-completely) perfect. The reverse follows immediately from Proposition 2.25 and the fact that a J-complete graded B_0-complex is graded (J-completely) perfect if and only if it is concentrated in finitely many grading and each grading piece is (J-completely) perfect. □

Corollary 2.27. *Suppose that $B_\bullet$ is indexed by $\mathbb{N}$ and connective. Then $M_\bullet \in \mathcal{DG}_{J\text{-}comp}(B_\bullet)$ is graded (J-completely) finite projective if and only if it has bounded below grading and $\mathrm{Red}_B(M_\bullet) \in \mathcal{DG}_{J\text{-}comp}(B_\bullet)$ is graded (J-completely) finite projective. Moreover $M_\bullet$ is graded finite projective (resp. graded perfect) if and only if it is graded J-completely finite projective (resp. graded J-completely perfect).*

Proof. The equivalence of graded perfect and graded J-completely perfect follows from Proposition 2.22 and Corollary 2.26. Among all four variants of graded finite projective, the condition of being graded finite projective is the strongest, and the condition of having bounded below grading with reduction being graded J-completely finite projective is the weakest, it suffices to know that the latter implies the former. Using Proposition 2.22 again, we see that $\mathrm{Red}_B(M_\bullet)$ being graded J-completely finite projective is equivalent to it being graded finite projective. Combining with Proposition 2.25, we are done since any extension $N_1 \otimes_{B_0} B_\bullet \to M_\bullet \to N_2 \otimes_{B_0} B_\bullet$, where N_i's are finite projective B_0-modules placed in a single degree, must split. □

For later purposes, let us observe the following graded version of Nakayama lemma.

Lemma 2.28. *Assume $M_\bullet \in \mathcal{DG}_{J\text{-}comp}(B_\bullet)$ has bounded below grading.*

(1) Assume $M_\bullet \neq 0$ and let c be the smallest integer n such that $M_n \neq 0$. Then $\mathrm{Red}_{<c,B}(M_\bullet) = 0$, and the natural map $M_c \to \mathrm{Red}_{c,B}(M_\bullet)$ is an isomorphism.
(2) We have that $M_{\leq m} = 0$ if and only if $\mathrm{Red}_{\leq m,B}(M_\bullet) = 0$.
(3) In particular, we have $M_\bullet = 0$ if and only if $\mathrm{Red}_B(M_\bullet) = 0$.

Proof. (2) and (3) follow from (1), below we prove (1). We take the graded tensor product of $M_\bullet$ with the following fiber sequence of $B_\bullet$-graded complexes

$$B_{\geq 1} \longrightarrow B_\bullet \longrightarrow B_0,$$

resulting a fiber sequence of graded B_0-complexes:

$$M_\bullet \otimes_{B_\bullet} B_{\geq 1} \to M_\bullet \to \mathrm{Red}_B(M_\bullet).$$

We now claim that $(M_\bullet \otimes_{B_\bullet} B_{\geq 1})$ has no degree $\leq c$ piece, which implies the first statement. To show the claim, we use the bar resolution of $M_\bullet \otimes_{B_\bullet} B_{\geq 1}$, which says that it can be computed by the colimit of the following simplicial diagram in the world of derived J-complete A-complexes:

$$M_\bullet \otimes_{B_\bullet} B_{\geq 1} \simeq \mathrm{colim}_{[n]\in\Delta}(M_\bullet \otimes_A B_\bullet \otimes_A \cdots \otimes_A B_\bullet \otimes_A B_{\geq 1}),$$

where the n-th term in the right hand side has n copies of $B_\bullet$ in the tensor product. Notice that for a given $[n] \in \Delta$, the degree i piece of the n-th term is isomorphic to the direct sum

$$\bigoplus_{u+w_1+\cdots+w_n+v=i} M_u \otimes_A B_{w_1} \otimes_A \cdots \otimes_A B_{w_n} \otimes_A B_v,$$

where each direct summand is zero unless $u \geq c$, $w_j \geq 0$ and $v \geq 1$. As a consequence, the deg $\leq c$ piece of the entire simplicial diagram vanishes. □

2.3. **Filtered homological algebra.** Throughout this subsection, let $\mathrm{Fil}^\bullet B$ be a *connective* derived J-complete (decreasing) filtered $\mathbb{E}_\infty$-A-algebras *indexed by* $\mathbb{N}$. Denote $B \coloneqq \mathrm{Fil}^0 B = \mathrm{Fil}^i B$ for any $i \leq 0$.

Definition 2.29. An object $\mathrm{Fil}^\bullet M \in \mathcal{DF}_{J\text{-comp}}(\mathrm{Fil}^\bullet B)$ is called *constant after i-th filtration* if the transition maps $\mathrm{Fil}^j M \to \mathrm{Fil}^{j-1} M$ are isomorphisms for any $j \leq i$. In such a situation, we have that $\mathrm{Fil}^j M \to \mathrm{Fil}^{-\infty} M \coloneqq \left(\mathrm{colim}_{i\to-\infty}\mathrm{Fil}^i M\right)^\wedge_J$ is an isomorphism for any $j \leq i$.

We say an object $\mathrm{Fil}^\bullet M \in \mathcal{DF}_{J\text{-comp}}(\mathrm{Fil}^\bullet B)$ is 0 *after i-th filtration* if it is constant after i-th filtration and $\mathrm{Fil}^{-\infty} M = 0$, or equivalently if $\mathrm{Fil}^j M = 0$ for any $j \leq i$.

We say a filtered object is *eventually constant* (resp. *eventually* 0) if it is constant (resp. 0) after i-th filtration for some i. We use $\mathcal{DF}^{\mathrm{ev.const}}_{J\text{-comp}}(\mathrm{Fil}^\bullet B)$ to denote the full subcategory spanned by eventually constant objects.

Below for a filtered algebra $\mathrm{Fil}^\bullet B$ that is indexed by $\mathbb{N}$, we shall view the quotient $\mathrm{Gr}^0 B$ as a filtered $\mathrm{Fil}^\bullet B$-algebra with its positive filtrations being zero. The following is a key technical result of this subsection.

Proposition 2.30. *Assume that every finite projective $\mathrm{Gr}^0 B$-module can be lifted to a finite projective B-module. Then for an object $\mathrm{Fil}^\bullet M \in \mathcal{DF}_{J\text{-comp}}(\mathrm{Fil}^\bullet B)$, the following four conditions are equivalent:*

(1) It is filtered perfect;
(2) It is filtered J-completely perfect;
(3) It is eventually constant with perfect $\mathrm{Fil}^{-\infty} M$ and the filtered base change $\mathrm{Fil}^\bullet M \otimes_{\mathrm{Fil}^\bullet B} \mathrm{Gr}^0 B$ is filtered perfect;
(4) It is eventually constant with J-completely perfect $\mathrm{Fil}^{-\infty} M$ and the filtered base change $\mathrm{Fil}^\bullet M \otimes_{\mathrm{Fil}^\bullet B} \mathrm{Gr}^0 B$ is filtered J-completely perfect.

Note that we are assuming $\mathrm{Fil}^\bullet B$ to be connective, hence it makes sense to talk about finite projective modules over B or $\mathrm{Gr}^0 B$. Moreover we remind readers that our $\otimes$ is always J-completely derived tensor product, so the filtered base change lives in $\mathcal{DF}_{J\text{-comp}}(\mathrm{Gr}^0 B)$.

It is clear that (1) implies (2), (2) implies (4). Moreover (3) and (4) are equivalent due to Proposition 2.22. So we just need to show that (3) implies (1). To that end, we need some preparations. First we notice that the filtered base change is compatible with taking graded pieces via graded base changes:

Lemma 2.31. *The following diagram commutes:*

$$\begin{tikzcd}
\mathcal{DF}_{J\text{-comp}}(\mathrm{Fil}^\bullet B) \arrow[r, "-\otimes_{\mathrm{Fil}^\bullet B}\mathrm{Gr}^0 B"] \arrow[d, "\mathrm{Gr}^\bullet"'] & \mathcal{DF}_{J\text{-comp}}(\mathrm{Gr}^0 B) \arrow[d, "\mathrm{Gr}^\bullet"] \\
\mathcal{DG}_{J\text{-comp}}(\mathrm{Gr}^\bullet B) \arrow[r, "-\otimes_{\mathrm{Gr}^\bullet B}\mathrm{Gr}^0 B"] & \mathcal{DG}_{J\text{-comp}}(\mathrm{Gr}^0 B).
\end{tikzcd}$$

Proof. It follows from the general fact that taking graded is symmetric monoidal, see [GL21, Lemma 3.10]. □

Lemma 2.32. *Let $\mathrm{Fil}^\bullet M \in \mathcal{DF}_{J\text{-comp}}(\mathrm{Fil}^\bullet B)$, we have:*

(1) It is constant after i-th filtration if and only if $\mathrm{Gr}^j M = 0$ for any $j \leq (i-1)$;
(2) It is constant after i-th filtration if and only if it is eventually constant and the filtered base change $\mathrm{Fil}^\bullet M \otimes_{\mathrm{Fil}^\bullet B} \mathrm{Gr}^0 B$ is constant after i-th filtration.
(3) The natural map
$$\mathrm{Fil}^{-\infty} M \otimes_B \mathrm{Gr}^0 B \to \mathrm{Fil}^{-\infty}(\mathrm{Fil}^\bullet M \otimes_{\mathrm{Fil}^\bullet B} \mathrm{Gr}^0 B)$$
is an isomorphism.
(4) Assume $\mathrm{Fil}^\bullet M$ is 0 after i-th filtration, then the filtered base change $\mathrm{Fil}^\bullet M \otimes_{\mathrm{Fil}^\bullet B} \mathrm{Gr}^0 B$ is 0 after i-th filtration.
(5) $\mathrm{Fil}^\bullet M = 0$ if and only if it is eventually 0 and the filtered base change $\mathrm{Fil}^\bullet M \otimes_{\mathrm{Fil}^\bullet B} \mathrm{Gr}^0 B = 0$.

Proof. (1): the transition map $\mathrm{Fil}^j M \to \mathrm{Fil}^{j-1} M$ is an isomorphism if and only if its cone, which by definition is $\mathrm{Gr}^{j-1} M$, is 0.

(2): Using (1), the statement follows from Lemma 2.28.(2) and Lemma 2.31.

(3): By derived Nakayama, we may reduce to the case where $J = 0$. In that case, it is a general fact that filtered base change commutes with taking underlying module of the factors and taking underlying ring of the tensor base.

(4): Let $\mathrm{Fil}^\bullet M \in \mathcal{DF}^{\text{ev.const}}_{I\text{-}comp}(I^\bullet A)$ and suppose it is 0 after i-th filtration. Then the filtered base change $\mathrm{Fil}^\bullet M \otimes_{\mathrm{Fil}^\bullet B} \mathrm{Gr}^0 B$ is:

- Constant after i-th filtration by (2);
- Eventually 0 by (3).

Therefore the filtered base change is 0 after i-th filtration.

(5): We only need to see the "if" part: by combining the condition with Lemma 2.28.(2) and Lemma 2.31, we see that $\mathrm{Gr}^\bullet M = 0$. Since $\mathrm{Fil}^\bullet M$ is eventually zero, we see that $\mathrm{Fil}^i M = \mathrm{Fil}^{-\infty} M = 0$. □

Lemma 2.33. *Let us denote the forgetful functor $\mathcal{D}(\mathrm{Gr}^0 B) \to \mathcal{D}(B)$ by ι_*. Assume $\mathrm{Fil}^\bullet M \in \mathcal{DF}_{J\text{-}comp}(\mathrm{Fil}^\bullet B)$ is 0 after i-th filtration, then the natural arrow $\mathrm{Fil}^{i+1} M \to \iota_*(\mathrm{Fil}^{i+1}(\mathrm{Fil}^\bullet M \otimes_{\mathrm{Fil}^\bullet B} \mathrm{Gr}^0 B))$ is an isomorphism.*

Proof. By the compatibility in Lemma 2.31, we have

$$\mathrm{Gr}^i(\mathrm{Fil}^\bullet M \otimes_{\mathrm{Fil}^\bullet B} \mathrm{Gr}^0 B) \cong \mathrm{Gr}^i(\mathrm{Gr}^\bullet M \otimes_{\mathrm{Gr}^\bullet B} \mathrm{Gr}^0 B).$$

Our claim now follows from Lemma 2.28.(1) and Lemma 2.32.(4). □

Using the above preparations, we can show that (3) implies (1) in Proposition 2.30 when $\mathrm{Fil}^{-\infty} M = 0$.

Proposition 2.34. *Assume that every finite projective $\mathrm{Gr}^0 B$-module can be lifted to a finite projective B-module. Then an object $\mathrm{Fil}^\bullet M \in \mathcal{DF}_{J\text{-}comp}(\mathrm{Fil}^\bullet B)$ is filtered perfect if*

(1) It is eventually 0; and

(2) the filtered base change $\mathrm{Fil}^\bullet M \otimes_{\mathrm{Fil}^\bullet B} \mathrm{Gr}^0 B$ is filtered perfect.

Proof. Without loss of generality, we may assume that $\mathrm{Fil}^\bullet M$ is 0 after 0-th filtration. By our assumption of filtered perfectness of $\mathrm{Fil}^\bullet M \otimes_{\mathrm{Fil}^\bullet B} \mathrm{Gr}^0 B =: \mathrm{Fil}^\bullet \overline{M}$, combined with Lemma 2.32.(4), we see that $\mathrm{Fil}^\bullet \overline{M}$ looks like

$$\left(\cdots \xrightarrow{=} 0 = \mathrm{Fil}^n \overline{M} \to \mathrm{Fil}^{n-1} \overline{M} \to \cdots \to \mathrm{Fil}^1 \overline{M} \to 0 = \mathrm{Fil}^0 \overline{M} \xrightarrow{=} \cdots\right),$$

where all $\mathrm{Fil}^i \overline{M} \in \mathrm{Perf}(\mathrm{Gr}^0 B)$. We shall prove by a double induction, the first layer of which is on the natural number n. When $n = 0$, by Lemma 2.32.(5) we see that $\mathrm{Fil}^\bullet M = 0$, hence trivially filtered perfect.

We need to prove the induction step, we claim that there exists a filtered perfect $\mathrm{Fil}^\bullet N \in \mathcal{DF}_{J\text{-comp}}(\mathrm{Fil}^\bullet B)$, which is 0 after 0-th filtration, together with a map $\mathrm{Fil}^\bullet N \to \mathrm{Fil}^\bullet M$ such that its cone $\mathrm{Fil}^\bullet C$ has reduction looking like

$$\left(\cdots \xrightarrow{=} 0 = \mathrm{Fil}^n \overline{C} \to \mathrm{Fil}^{n-1} \overline{C} \to \cdots \to 0 = \mathrm{Fil}^1 \overline{C} \xrightarrow{=} \cdots\right).$$

Granting the claim, then by shifting the filtration of $\mathrm{Fil}^\bullet C$, we see by induction that $\mathrm{Fil}^\bullet C$ is filtered perfect, therefore $\mathrm{Fil}^\bullet M$ is filtered perfect as well.

Next we prove the above claim, using induction on the length of tor-amplitude of $\mathrm{Fil}^1 \overline{M} \in \mathrm{Perf}(\mathrm{Gr}^0 B)$. Since we may perform cohomological shift of $\mathrm{Fil}^\bullet M$, let us assume the tor-amplitude of $\mathrm{Fil}^1 \overline{M} \in \mathrm{Perf}(\mathrm{Gr}^0 B)$ is in $[0, m]$. The base case is $m = 0$, namely when $\mathrm{Fil}^1 \overline{M}$ is a finite projective $\mathrm{Gr}^0 B$-module. Our assumption says that there exists a finite projective B-module V with $V \otimes_B \mathrm{Gr}^0 B \cong \mathrm{Fil}^1 \overline{M}$. Applying Lemma 2.33 with $i = 0$, we see that there is a map $V \to \mathrm{Fil}^1 M \cong \iota_* \mathrm{Fil}^1 \overline{M}$ in $D(B)$ inducing $\iota^* V = V \otimes_B \mathrm{Gr}^0 B \xrightarrow{\cong} \mathrm{Fil}^1 \overline{M}$. This implies that there is a map from the filtered perfect

$$\mathrm{Fil}^\bullet N' := (\cdots \xrightarrow{=} 0 = \mathrm{Fil}^2 \to V = \mathrm{Fil}^1 \to 0 = \mathrm{Fil}^0 \xrightarrow{=} \cdots) \longrightarrow \mathrm{Fil}^\bullet M$$

in $\mathcal{DF}_{J\text{-comp}}(B)$ where the target is viewed as an object via forgetful functor $\mathcal{DF}_{J\text{-comp}}(\mathrm{Fil}^\bullet B) \to \mathcal{DF}_{J\text{-comp}}(B)$. By adjunction, we get a map $\mathrm{Fil}^\bullet N := \mathrm{Fil}^\bullet N' \otimes_B (\mathrm{Fil}^\bullet B) \to \mathrm{Fil}^\bullet M$. Its reduction is given by

$$\mathrm{Fil}^\bullet \overline{N} = (\cdots \xrightarrow{=} 0 = \mathrm{Fil}^2 \to V \otimes_B \mathrm{Gr}^0 B = \mathrm{Fil}^1 \to 0 = \mathrm{Fil}^0 \xrightarrow{=} \cdots) \to \mathrm{Fil}^\bullet \overline{M}$$

which in Fil^1 is exactly the isomorphism $\iota^* V = V \otimes_B \mathrm{Gr}^0 B \xrightarrow{\cong} \mathrm{Fil}^1 \overline{M}$. This proves the claim for the case $m = 0$.

Lastly, we prove the case for general m: Choose a surjection from a finite free $\pi_0(\mathrm{Gr}^0 B)^{\oplus r} \twoheadrightarrow H^m(\mathrm{Fil}^1\overline{M})$, which can be lifted to a map $(\mathrm{Gr}^0 B)^{\oplus r}[-m] \to \mathrm{Fil}^1\overline{M}$ whose cone has tor-amplitude $[0, m-1]$. Repeat the previous argument, we have a map in $\mathcal{DF}_{J\text{-comp}}(\mathrm{Fil}^\bullet B)$:

$$\mathrm{Fil}^\bullet L := (\cdots \xrightarrow{=} 0 = \mathrm{Fil}^2 \to B^{\oplus r}[-m] = \mathrm{Fil}^1 \to 0 = \mathrm{Fil}^0 \xrightarrow{=} \cdots) \otimes_B \mathrm{Fil}^\bullet B \longrightarrow \mathrm{Fil}^\bullet M,$$

whose cone has its reduction of the form

$$\left(\cdots \xrightarrow{=} 0 = \mathrm{Fil}^n\overline{M} \to \mathrm{Fil}^{n-1}\overline{M} \to \cdots \to \mathrm{Cone}(\mathrm{Gr}^0 B^{\oplus r}[-m] \to \mathrm{Fil}^1\overline{M}) \to 0 = \mathrm{Fil}^0\overline{M} \xrightarrow{=} \cdots\right).$$

Since $\mathrm{Fil}^\bullet L$ is 0 after 0-th filtration, filtered perfect, and the tor-amplitude of $\mathrm{Fil}^1\overline{(M/L)}$ is now $[0, m-1]$, we are done by induction. □

We are finally ready to prove Proposition 2.30.

Proof of Proposition 2.30. According to the discussion after its statement, it suffices to show that (3) implies (1). Without loss of generality, we assume that $\mathrm{Fil}^\bullet M$ is constant after 0-th filtration. Then $\mathrm{Fil}^0 M \cong \mathrm{Fil}^{-\infty} M$ is perfect over B. Similar to the argument as in the third paragraph of the proof of Proposition 2.34, we have a map $(\cdots \to \mathrm{Fil}^1 M = 0 \to \mathrm{Fil}^0 M \xrightarrow{=} \mathrm{Fil}^{-1} M \xrightarrow{=} \cdots) \otimes_B \mathrm{Fil}^\bullet B \to \mathrm{Fil}^\bullet M$ which is an isomorphism in $\mathrm{Fil}^{\geq 0}$. Hence its cone is 0 after 0-th filtration and has filtered perfect reduction. By Proposition 2.34 we see that this cone is filtered perfect, hence we win. □

We have the following analog for filtered finite projective modules.

Proposition 2.35. *Assume that for every finite projective B-module P, every direct summand $\overline{P_1} \subset \overline{P} := P \otimes_B \mathrm{Gr}^0 B$ can be lifted to a direct summand $P_1 \subset P$. Then for an object $\mathrm{Fil}^\bullet M \in \mathcal{DF}_{J\text{-comp}}(\mathrm{Fil}^\bullet B)$, the following four conditions are equivalent:*

1. *It is a finite direct sum of $N_i \otimes_B \mathrm{Fil}^\bullet B\langle n_i\rangle$ where each N_i is a finite projective B-module.*
2. *It is filtered finite projective;*
3. *It is filtered J-completely finite projective;*
4. *It is eventually constant with finite projective $\mathrm{Fil}^{-\infty} M$ and the filtered base change $\mathrm{Fil}^\bullet M \otimes_{\mathrm{Fil}^\bullet B} \mathrm{Gr}^0 B$ is filtered finite projective;*
5. *It is eventually constant with J-completely finite projective $\mathrm{Fil}^{-\infty} M$ and the filtered base change $\mathrm{Fil}^\bullet M \otimes_{\mathrm{Fil}^\bullet B} \mathrm{Gr}^0 B$ is filtered J-completely finite projective.*

Proof. Similar to the proof of Proposition 2.30, it is easy to see that (1) implies (2), (2) implies (3), and (3) implies (5). Moreover, for any connective derived J-complete $\mathbb{E}_\infty$-A-algebra R, an object $\mathrm{Fil}^\bullet M \in \mathcal{DF}_{J\text{-comp}}(R)$ is filtered (resp. J-completely) filtered finite projective if and only if the filtration is complete and exhaustive with finitely many nonzero graded pieces and each graded piece is finite projective (resp. J-completely finite projective). Therefore we see that (4) and (5) are again equivalent, by applying Proposition 2.22 to $R = B$ and $R = \mathrm{Gr}^0 B$. Below let us show that (4) implies (1).

Without loss of generality, let us assume that $\mathrm{Fil}^\bullet M$ is constant after 0-th filtration. By Lemma 2.32.(2), we know that $\mathrm{Fil}^\bullet M \otimes_{\mathrm{Fil}^\bullet B} \mathrm{Gr}^0 B =: \mathrm{Fil}^\bullet\overline{M}$ is also constant after 0-th filtration, therefore it looks like

$$\left(\cdots \xrightarrow{=} 0 = \mathrm{Fil}^n\overline{M} \to \mathrm{Fil}^{n-1}\overline{M} \to \cdots \to \mathrm{Fil}^1\overline{M} \to \mathrm{Fil}^0\overline{M} \xrightarrow{=} \cdots\right)$$

where each $\mathrm{Fil}^i\overline{M} \to \mathrm{Fil}^{i-1}\overline{M}$ is a direct summand of finite projective $\mathrm{Gr}^0 B$-modules. Let us perform an induction on n, the base case being $n = 1$. When $n = 1$, we consider the cone C of $\mathrm{Fil}^0 M \otimes_B \mathrm{Fil}^\bullet B \to \mathrm{Fil}^\bullet M$ induced by the natural map $\mathrm{Fil}^0 M \xrightarrow{=} \mathrm{Fil}^0 M$. The cone is 0 after 0-th filtration, and its filtered base change

$$\mathrm{Fil}^\bullet\overline{C} \cong \mathrm{Cone}\Big((\cdots \xrightarrow{=} 0 = \mathrm{Fil}^1 \to \mathrm{Fil}^0 = \mathrm{Fil}^0 M \otimes_B \mathrm{Gr}^0 B \xrightarrow{=} \cdots) \to \mathrm{Fil}^\bullet\overline{M}\Big)$$

is 0 by Lemma 2.32.(3). Therefore we see that $C = 0$, by Lemma 2.32.(5). Since $\mathrm{Fil}^{-\infty} M = \mathrm{Fil}^0 M$ is a finite projective B-module, we get that $\mathrm{Fil}^0 M \otimes_B \mathrm{Fil}^\bullet B = \mathrm{Fil}^\bullet M$ is also filtered finite projective over $\mathrm{Fil}^\bullet B$.

In general, choose a splitting $\mathrm{Fil}^0\overline{M} \cong \mathrm{Fil}^1\overline{M} \oplus \mathrm{Gr}^0\overline{M}$ and lift it to $\mathrm{Fil}^0 M \cong N_1 \oplus N_2$. Now we consider the cone C of $N_2 \otimes_B \mathrm{Fil}^\bullet B \to \mathrm{Fil}^\bullet M$ induced by the decomposition in Fil^0. It is constant after 0-th step and this constant is N_1 which is a finite projective B-module. Moreover, by construction, its filtered base change $\mathrm{Fil}^\bullet\overline{C}$ has constant filtration after the 1-st filtration and the induced map $\mathrm{Fil}^{\geq 1}\overline{M} \to \mathrm{Fil}^{\geq 1}\overline{C}$ is an isomorphism. Therefore, by induction, we see that C is of the form described in (1). Lastly, we only need to know that a triangle in $\mathcal{DF}_{J\text{-comp}}(\mathrm{Fil}^\bullet B)$ of the form

$$\mathrm{Fil}^\bullet M' \to \mathrm{Fil}^\bullet M \to V \otimes_B \mathrm{Fil}^\bullet B\langle n\rangle$$

must split if the first term is connective and V is finite projective over B: Indeed, the first condition implies $\pi_0(\mathrm{Fil}^{-n} M) \twoheadrightarrow \pi_0(V)$ is surjective, hence there is a splitting $s\colon V \to \mathrm{Fil}^{-n} M$ of the map of B-complexes $\mathrm{Fil}^{-n} M \to V$, therefore we get the desired filtered splitting $s \otimes_B \mathrm{Fil}^\bullet B$. □

2.4. **Descent statements.** In this subsection, we establish descent statements in the filtered and derived J-complete setting.

Digression 2.36. Recall that a map of connective $\mathbb{E}_1$-ring $R \to S$ is said to be (faithfully) flat if $\pi_0(R) \to \pi_0(S)$ is (faithfully) flat and the induced maps $\pi_i(R) \otimes_{\pi_0(R)} \pi_0(S) \to \pi_i(S)$ are isomorphisms for all i, see [Lur17, Definition 7.2.2.10]. In [Lur17, Theorem 7.2.2.15] one can find many equivalent conditions characterizing flatness.

Definition 2.37. We call a map of connective derived J-complete $\mathbb{E}_\infty$-A-algebra $R \to S$ to be *J-completely (faithfully) flat* if its base change $A/J \otimes_A R \to A/J \otimes_A S$ is so.

By [Lur18, Proposition 2.7.3.2], for any connective R' with a factorization $R \to R' \to A/J \otimes_A R$ such that $\pi_0(R') \to \pi_0(R)/J$ has nilpotent kernel, the J-completely (faithful) flatness is equivalent to having the base change $R' \to R' \otimes_R S$ being (faithfully) flat. In particular, the above definition only depends on $V(J) \subset \mathrm{Spec}(A)$.

Proposition 2.38. *Let $\mathrm{Fil}^\bullet B \to \mathrm{Fil}^\bullet B^{(0)}$ be a map of connective derived J-complete (decreasing) filtered $\mathbb{E}_\infty$-A-algebras indexed by $\mathbb{N}$, and form the i-th Cech nerve $\mathrm{Fil}^\bullet B^{(i)}$ for $[i] \in \Delta$. Assume that the map of Rees's construction $(\bigoplus_{m\in\mathbb{Z}} \mathrm{Fil}^m B) \to (\bigoplus_{m\in\mathbb{Z}} \mathrm{Fil}^m B^{(0)})$ is J-completely faithfully flat. Then the following natural functors induced by base change:*

(1) $\mathcal{DF}_{J\text{-comp}}(\mathrm{Fil}^\bullet B) \to \lim_{[i]\in\Delta} \mathcal{DF}_{J\text{-comp}}(\mathrm{Fil}^\bullet B^{(i)})$;
(2) $\mathcal{DF}^{\mathrm{perf}}_{J\text{-comp}}(\mathrm{Fil}^\bullet B) \to \lim_{[i]\in\Delta} \mathcal{DF}^{\mathrm{perf}}_{J\text{-comp}}(\mathrm{Fil}^\bullet B^{(i)})$; and
(3) $\mathcal{DF}^{\mathrm{vect}}_{J\text{-comp}}(\mathrm{Fil}^\bullet B) \to \lim_{[i]\in\Delta} \mathcal{DF}^{\mathrm{vect}}_{J\text{-comp}}(\mathrm{Fil}^\bullet B^{(i)})$

are all equivalences. Here vect *and* perf *denote the full subcategory of filtered J-completely finite projective and filtered J-completely perfect objects respectively.*

Proof. Let us show (1) first. It is equivalent to showing:

(1) For any $\mathrm{Fil}^\bullet M \in \mathcal{DF}_{J\text{-comp}}(\mathrm{Fil}^\bullet B)$, the natural arrow $\mathrm{Fil}^\bullet M \to \lim_{[i]\in\Delta} \mathrm{Fil}^\bullet B^{(i)} \otimes_{\mathrm{Fil}^\bullet B} \mathrm{Fil}^\bullet M$ is an equivalence; and
(2) For any $\mathrm{Fil}^\bullet M^{(i)} \in \lim_{[i]\in\Delta} \mathcal{DF}_{J\text{-comp}}(\mathrm{Fil}^\bullet B^{(i)})$, the natural arrow $\mathrm{Fil}^\bullet B^{(m)} \otimes_{\mathrm{Fil}^\bullet B} (\lim_{[i]\in\Delta} \mathrm{Fil}^\bullet M^{(i)}) \to \mathrm{Fil}^\bullet M^{(m)}$ is an equivalence for all $[m] \in \Delta$.

By derived Nakayama, it suffices to show the above are equivalences after applying $\mathrm{Kos}(A; f_i) \otimes_A -$, where $\{f_i\}$ is a chosen finite set of generators of J. Since $\mathrm{Kos}(A; f_i)$ is a perfect complex over A, the functor commutes with limit. Therefore we are reduced to the case where $J = 0$. Since Rees's construction commutes with cosimplicial limit, and *when* $J = 0$ it is symmetric monoidal: Namely, it sends the filtered tensor product to the usual tensor product, we are finally reduced to the usual fpqc descent [Lur18, Corollary D.6.3.3].

Next we show (2) and (3): Given (1), it suffices to show that $\mathrm{Fil}^\bullet B^{(0)} \otimes_{\mathrm{Fil}^\bullet B} \mathrm{Fil}^\bullet M$ being filtered J-completely finite projective (resp. perfect) implies that the original $\mathrm{Fil}^\bullet M$ is filtered J-completely finite projective (resp. perfect). Using Lemma 2.9 and Proposition 2.17, we are reduced to the usual equivalence of being finite projective (resp. perfect) before and after the faithfully flat base change [BL22b, Proposition 2.8.4.2]. □

We are also interested in descent even when the filtered rings involved are *not* connective. For that purpose. we shall use the notion of "descendability" by Mathew, see [Mat16, Definition 3.18]. Below let us collect some useful facts about descendable maps between $\mathbb{E}_\infty$-rings.

Digression 2.39 ([Mat16, §3], see also [BS17, §11.2])**.** Descendability is preserved under base change, see [Mat16, Corollary 3.21]. If $R \to S$ is a descendable map between $\mathbb{E}_\infty$-rings, and let $S^{(\bullet)}$ be the Cech nerve, then [Mat16, Proposition 3.22] gives the descent statement: the natural functor induced by base change $\mathcal{D}(R) \to \lim_{[i]\in\Delta} \mathcal{D}(S^{(\bullet)})$ is an equivalence. By [Mat16, Proposition 3.28], we see that an R-complex M is perfect if and only if $S \otimes_R M \in \mathcal{D}(S)$ is perfect.

In fact, Mathew works with general "*stable homotopy theory*", examples of which include the $\mathcal{DF}_{J\text{-comp}}(A)$ that we have been working on throughout this section. Below we explicate the consequences that one can draw out of Mathew's theory of descendability, when specialized to $\mathcal{DF}_{J\text{-comp}}(A)$.

Proposition 2.40. *Let* $\mathrm{Fil}^\bullet B \to \mathrm{Fil}^\bullet B^{(0)}$ *be a* descendable *map of derived J-complete (decreasing) filtered $\mathbb{E}_\infty$-A-algebras indexed by $\mathbb{Z}$, and form the i-th Cech nerve* $\mathrm{Fil}^\bullet B^{(i)}$ *for $[i] \in \Delta$. Then the following natural functors induced by base change:*

(1) $\mathcal{DF}_{J\text{-}comp}(\mathrm{Fil}^\bullet B) \to \lim_{[i]\in\Delta} \mathcal{DF}_{J\text{-}comp}(\mathrm{Fil}^\bullet B^{(i)})$*; and*

(2) $\mathcal{DF}^{\mathrm{perf}}_{J\text{-}comp}(\mathrm{Fil}^\bullet B) \to \lim_{[i]\in\Delta} \mathcal{DF}^{\mathrm{perf}}_{J\text{-}comp}(\mathrm{Fil}^\bullet B^{(i)})$*;*

are both equivalences.

Proof. The first statement follows from [Mat16, Proposition 3.22]. For (2), we need to know that $\mathrm{Fil}^\bullet M$ is filtered J-completely perfect as soon as $\mathrm{Fil}^\bullet B^{(0)} \otimes_{\mathrm{Fil}^\bullet B} \mathrm{Fil}^\bullet M$ is filtered J-completely perfect. By definition, we need to perform $A/J \otimes_A -$ and ask filtered perfectness, hence we are reduced to the case of $J = 0$. In this case, the Rees's construction is symmetric monoidal. By [Mat16, Corollary 3.21], we see that the map of Rees's construction $(\bigoplus_{m\in\mathbb{Z}} \mathrm{Fil}^m B) \to (\bigoplus_{m\in\mathbb{Z}} \mathrm{Fil}^m B^{(0)})$ is descendable in $\mathcal{D}(A)$. The perfectness of $\mathrm{Fil}^\bullet M$ now follows from the combination of Proposition 2.17 and [Mat16, Proposition 3.28]. □

3. Absolute prismatic F-crystals and prismatic F-gauges

In this section, we recall the notion of absolute prismatic F-crystals and absolute prismatic F-gauges, and consider their relation.

3.1. The coherent prismatic (F-)crystals.

Let X be a quasi-syntomic p-adic formal scheme over $\mathcal{O}_K$. Recall from [BS22] and [BS23] that we can attach to X the following ringed sites:

Definition 3.1. The *(absolute) prismatic site of* X is defined as the opposite category $X_{\mathbb{\Delta}}$ of bounded prisms (A, I) over X with the (p, I)-complete flat topology, and is equipped with a sheaf of rings and a sheaf of ideals

$$\mathcal{O}_{\mathbb{\Delta}} := (A, I) \longmapsto A;\ \mathcal{I}_{\mathbb{\Delta}} := (A, I) \longmapsto I.$$

The prismatic structure sheaf $\mathcal{O}_{\mathbb{\Delta}}$ admits a natural Frobenius action which lifts the usual Frobenius on its reduction mod p.

Definition 3.2. The *quasiregular semiperfectoid site of* X is the opposite category X_{qrsp} of quasiregular semiperfectoid algebras which are quasi-syntomic over X, equipped with the quasi-syntomic topology. It comes with a sheaf of prisms:

$$(\mathbb{\Delta}_\bullet, I_\bullet) := S \longmapsto (\mathbb{\Delta}_S, I_{\mathbb{\Delta}_S})$$

sending $S \in X_{\mathrm{qrsp}}$ to the initial prism of $\mathrm{Spf}(S)_{\mathbb{\Delta}}$ as in [BS22, Prop. 7.2]. The fact that they are quasi-syntomic sheaves follows from the theory of Nygaard filtration and quasi-syntomic descent for conjugate filtrations, see [BS22, §12]. Notice that for every S, the ideal $I_{\mathbb{\Delta}_S}$ is non-canonically principal. The Frobenius endomorphism of prisms induces a natural endomorphism φ on $\mathbb{\Delta}_\bullet$.

The above allows us to define various notions of coefficients associated to X. Below we let $* \in \{\mathrm{vect}, \mathrm{perf}, \emptyset\}$ to denote the corresponding category with coefficients in either vector bundles, perfect complexes or general complexes.

Definition 3.3. Denote by (F-)Crys$^*(X)$ the category of *prismatic* (F-)*crystals* (in either vector bundles or perfect complexes or general complexes) over X, in the sense of [BS23, Definition 4.1].

Concretely, an F-crystal consists of a vector bundle/perfect complex/general complex $\mathcal{E}$ over the prismatic site X_{Δ} together with a Frobenius-isogeny, namely an $\mathcal{O}_{\Delta}$-linear isomorphism

$$\varphi_{\mathcal{E}} : (\varphi^*_{\mathcal{O}_{\Delta}}\mathcal{E})[1/\mathcal{I}] \simeq \mathcal{E}[1/\mathcal{I}].$$

By [BS23, Proposition 2.13 and Proposition 2.14], one also has the following descriptions when $* \in \{\mathrm{vect}, \mathrm{perf}\}$

$$\text{(F-)Crys}^*(X) \simeq \lim_{S\in X_{\mathrm{qrsp}}} \text{(F-)Crys}^*(\mathrm{Spf}(S)).$$

Among objects in X_{Δ}, the following type of prisms plays a very useful role.

Definition 3.4. Assume X is smooth over $\mathcal{O}_K$. A *Breuil–Kisin* prism (A, I) in X_{Δ} is a prism such that $\mathrm{Spf}(\overline{A}) \to X$ is an open immersion (c.f. [DLMS22, Example 3.4] and [GR22, proof of Theorem 5.10]).

Below we show that such A is always with a faithfully flat Frobenius endomorphism $\phi_A : A \to A$. We have the following simple description of Breuil–Kisin prisms.

Lemma 3.5. *Let (A, I) be a Breuil–Kisin prism over X as above, and let $\widetilde{R}$ be a smooth lift of $\overline{A}/\pi$ over W.*

1. *If $e > 1$, then there is an isomorphism of pairs $(A, I) \simeq (\widetilde{R}[\![u]\!], E(u))$, where $E(u)$ is the Eisenstein polynomial of a uniformizer π.*
2. *If $e = 1$, assuming $\mathrm{Spf}(A)$ is oriented, then we have an isomorphism of pairs $(A, I) \simeq (\widetilde{R}[\![u]\!], u - p)$.*

Proof. By deformation theory, we can lift the isomorphism $\widetilde{R}/p \simeq \overline{A}/\pi$ to an isomorphism $\widetilde{R} \otimes_W \mathcal{O}_K \xrightarrow{\simeq} \overline{A}$. Again by deformation theory, we can lift the morphism $\widetilde{R} \to \overline{A}$ further to $\widetilde{R} \to A$.

Now we can start the proof. Assume $e > 1$, pick a uniformizer $\pi \in \mathcal{O}_K \subset \overline{A}$, lift it to an element $\widetilde{\pi} \in A$. Noticing that $\widetilde{\pi}$ is topologically nilpotent in A, we get a map $\widetilde{R}[\![u]\!] \to A$, sending u to $\widetilde{\pi}$. Now we observe that the Eisenstein polynomial $E(u)$ is sent to zero in $\overline{A}$, hence $E(u)$ necessarily lands in $I \subset A$, and the source is $E(u)$-adically complete whereas the target is I-adically complete. Therefore to finish our proof, we just need to check that $E(\widetilde{\pi})$ generates I in A. Writing $E(\widetilde{\pi}) = \widetilde{\pi}^n + p \cdot f$, where f is a unit, and let us compute

$$p\delta(E(\widetilde{\pi})) = \left(\widetilde{\pi}^p + p\delta(\widetilde{\pi})\right)^n + p \cdot \varphi(f) - E(\widetilde{\pi})^p.$$

Using the fact that $n > 1$, one can check that the right hand side is $p \cdot \varphi(f) + p \cdot g$ where $g \in (p, \widetilde{\pi})$. Since A is p-torsion free, this implies that $\delta(E(\widetilde{\pi}))$ is a unit. Therefore [BS22, Lemma 2.24] gives $E(\widetilde{\pi}) = I$.

If instead $e = 1$ and (A, I) is oriented. Let us choose a generator d of I. Then we get an isomorphism $\widetilde{R}[\![\widetilde{u}]\!] \simeq A$ by sending $\widetilde{u}$ to d. Now we just let $u = \widetilde{u} + p$. □

Corollary 3.6. *Let (A, I) be a Breuil–Kisin prism over X as above, then A is a p-torsionfree regular ring and the Frobenius endomorphism $\phi_A : A \to A$ is quasi-syntomic, finite, and faithfully flat.*

Proof. The fact that A is p-torsionfree and regular follows immediately from the concrete description of A in Lemma 3.5. Also from the description, we see that the Frobenius on A/p is finite and lci, therefore its lift is also finite and quasi-syntomic thanks to p-completeness of A. Using regularity, finiteness, and miracle flatness [Sta23, Tag 00R4], we get that ϕ_A is flat. Faithfulness follows from p-completeness: the induced map on spectrum is a homeomorphism on the locus $V(p)$ which contains all closed points by p-completeness, noticing that flat map is open we see that the induced map on spectrum has to be surjective. □

If $X = \mathrm{Spf}(R)$ is further assumed to be affine, then we can find such an A so that (A, I) covers the final object in $Shv(X_{\Delta})$. It is because with mild assumptions, we can construct absolute product of any prism with Breuil–Kisin prisms in X_{Δ}, and they enjoy good complete flatness properties.

Proposition 3.7. *Assume X is separated and smooth over $\mathrm{Spf}(\mathcal{O}_K)$, let (A, I) be a Breuil–Kisin prism in X_{Δ} and let (A', I') be a prism in X_{Δ}. Then their product in $Shv(X_{\Delta})$ exists and is given by a prism (C, K). It has the following properties:*

1. *The map $(A', I') \to (C, K = I'C)$ is (p, I')-completely flat;*
2. *If $\mathrm{Im}(\mathrm{Spf}(\overline{A'}) \to X) \subset \mathrm{Spf}(\overline{A})$, then the map above is (p, I')-completely faithfully flat;*
3. *If the map $\mathrm{Spf}(\overline{A'}) \to X$ is p-completely flat, then the map $(A, I) \to (C, K = IC)$ is (p, I)-completely flat.*

(4) *If in (3) we assume furthermore that* $\mathrm{Spf}(\overline{A}) \subset \mathrm{Im}(\mathrm{Spf}(\overline{A'}) \to X)$*, then the map above is* (p, I)*-completely faithfully flat.*

Consequently, if $X = \mathrm{Spf}(R)$ *is affine, then we can find a Breuil–Kisin prism* (A, I) *which covers the final object* $* \in Shv(X_{\Delta})$.

This Proposition is certainly well-known to experts, for the convenience of the reader we spell out the proof in detail. Here we temporarily abuse the notation of K.

Proof. Let us construct the prism (C, K) first: it is supposed to be the absolute pushout of (A, I) and (A', I') in the category of prisms in X_{Δ}. Unwinding definition, we see that (C, K) is the initial object of prisms (D, L) receiving map from both (A, I) and (A', I') with the property that the induced map $\overline{A}\widehat{\otimes}_{W(k)}\overline{A'} \to D/L$ factors through $\overline{A}\widehat{\otimes}_{\mathcal{O}_X}\overline{A'}$. Here we make two remarks:

- For any prism $(D, L) \in X_{\Delta}$, using the fact that D/L^n are p-complete, we see that the map $W(k) \to \mathcal{O}_X \to D/L$ lifts uniquely to a map $W(k) \to D$ of δ-rings. This explains why the first tensor product above is over $W(k)$.
- Since X was assumed to be separated, the (completed) tensor product $\overline{A}\widehat{\otimes}_{\mathcal{O}_X}\overline{A'}$ is given by a (derived) p-complete ring corresponding to the formal scheme $\mathrm{Spf}(\overline{A}) \times_X \mathrm{Spf}(\overline{A'})$.

Back to the construction of (C, K), from the above description we see that it is necessarily given by the delta envelope of (B, J), whose meaning we shall explain below: Here $B \coloneqq A\widehat{\otimes^L}_{W(k)}A'$ denotes the derived $(p, 1 \otimes I', I \otimes 1)$-completed tensor product, and one checks that it is concentrated in degree 0 and is a (p, I')-completely flat δ-A'-algebra. Indeed if we derived mod B by $1 \otimes I'$, it becomes the derived (p, I)-completed tensor product $A\widehat{\otimes^L}_{W(k)}\overline{A'}$ which is seen to be a p-completely flat $\overline{A'}$-algebra due to Lemma 3.5. Let J denote the kernel ideal of the map $B \to \overline{A}\widehat{\otimes}_{\mathcal{O}_X}\overline{A'}$. Below we shall check that our ideal J meets the regularity condition in [BS22, Proposition 3.13]. Granting this the cited proposition constructs the prismatic envelope and moreover checks that it satisfies the property (1). Now let us check the regularity condition. Due to the Zariski locality of the regularity condition, we may assume that A is given by the concrete description in Lemma 3.5. Now we see that the ideal $J/(1 \otimes I')$ is Zariski locally on $\mathrm{Spf}(B/(1 \otimes I'))$ given by $u - 1 \otimes \pi$ and (the lift of) a regular sequence corresponding to the kernel of $\widetilde{R}\widehat{\otimes}_{W(k)}\overline{A'} \twoheadrightarrow \overline{A}\widehat{\otimes}_{\mathcal{O}_X}\overline{A'}$, where both completed tensor product are p-completed tensor product. The said kernel is Zariski locally a regular sequence because $\widetilde{R}$ is p-completely smooth $W(k)$-algebra and $\overline{A}$ defines an open of X, so the surjection is from a p-completely smooth $\overline{A'}$-algebra to a p-completely Zariski open $\overline{A'}$-algebra, which is lci mod p.

By the above paragraph, we have constructed (C, K) and it satisfies the properties listed in [BS22, Proposition 3.13]. In particular, it satisfies (1) of our proposition. Let us check property (2): we need to utilize the perfection $(A_\infty, I_\infty) \coloneqq (A, I)_{\mathrm{perf}}$ of (A, I) introduced in [BS22, Lemma 3.9]. Now Corollary 3.6 says that the Frobenius ϕ_A is quasi-syntomic and faithfully flat, by construction spelled out in loc. cit. we see that $(A, I) \to (A_\infty, I_\infty)$ is also quasi-syntomic and (p, I)-completely faithfully flat. Let us stare at the following diagram:

$$\begin{tikzcd}
(A', I') \arrow[r] & (C, K) \arrow[r] & (E, M) \\
& (A, I) \arrow[u] \arrow[r] & (A_\infty, I_\infty) \arrow[u]
\end{tikzcd}$$

where the square is a (p, I)-completely pushing out defining the prism (E, M). By the above discussion, we know the rightward arrows in this square are (p, I)-completely faithfully flat. Therefore we reduce our problem to checking the composite $(A', I') \to (E, M)$ is (p, I')-completely faithfully flat. To that end, let us study (E, M) from a slightly different perspective: By how it is defined, we see that (E, M) is the absolute pushout of (A_∞, I_∞) and (A', I') in the category of prisms in X_{Δ}. Since perfect prisms are initial prisms of their reductions ([BS22, Theorem 3.10]), we see that (E, M) is the initial object in $\big(S/A'\big)_{\Delta}$, where $S \coloneqq \overline{A'}\widehat{\otimes}_{\overline{A}}\overline{A_\infty}$. This is where we use the assumption that $\mathrm{Im}(\mathrm{Spf}(\overline{A'}) \to X) \subset \mathrm{Spf}(\overline{A})$. Now we check that S is a large quasi-syntomic p-completely faithfully flat $\overline{A'}$-algebra (see [BS22, Definition 15.1] for the meaning of being "large"): It is the p-complete base change of a quasi-syntomic p-completely faithfully flat morphism; as for largeness, just notice that any perfectoid algebra is large (being a quotient of the Witt vector of a perfect ring), hence so are their base changes. By [BS22, Theorem 15.2], we learn that E is given by the derived prismatic cohomology $\Delta_{S/A'}$ explained in [BS22, Construction 7.6]. Recall that we wanted to show $\overline{A'} \to \bar{E}$ is p-completely faithfully flat, the map factors through the p-completely faithfully flat map $\overline{A'} \to S$, hence it suffices to show $S \to \bar{E}$ is p-completely faithfully flat. To that end, one just notices that this map is the 0-th conjugate filtration on the derived Hodge–Tate cohomology $\bar{E} = \overline{\Delta}_{S/A'}$, and the higher graded pieces of the conjugate filtration are p-completely flat S-modules: indeed they are given by $\Gamma^{\bullet}_S(\mathbb{L}_{S/A'}[-1])^{\wedge}$ and the shifted cotangent complex $\mathbb{L}_{S/A'}[-1]$ is a p-completely flat S-module (c.f. the discussion right after [BS22, Definition 15.1]).

The proof of (3) is fairly easy: it is equivalent to asking the map $\overline{A} \to \overline{C}$ to be p-completely flat. We stare at the following commutative diagram:

$$\begin{tikzcd} \mathrm{Spf}(\overline{C}) \arrow[r] \arrow[d] & \mathrm{Spf}(\overline{A}) \arrow[d] \\ \mathrm{Spf}(\overline{A'}) \arrow[r] & X. \end{tikzcd}$$

Our assumption, together with (1) above, implies that the map $\mathrm{Spf}(\overline{C}) \to X$ is p-completely flat. Since the map $\mathrm{Spf}(\overline{A}) \to X$ is an open immersion, we get what we want.

As for (4): since X is separated, the preimage of $\mathrm{Spf}(\overline{A}) \subset X$ defines an affine open in $\mathrm{Spf}(\overline{A'})$ which is also an affine open $U \subset \mathrm{Spf}(A')$, we may replace A' by $\mathcal{O}_{\mathrm{Spf}(A')}(U)$ and replace X by $\mathrm{Spf}(\overline{A})$ without changing (C, K). Now we are reduced to the case where $X = \mathrm{Spf}(\overline{A}) = \mathrm{Im}(\mathrm{Spf}(A'))$. Looking again at the above diagram, we see that $\mathrm{Spf}(\overline{C}) \to X = \mathrm{Spf}(\overline{A})$ is p-completely faithfully flat due to (2) and (3). This finishes the proof of properties (1)-(4) listed in this Proposition.

For the last sentence, simply notice that when X is affine, one can find a Breuil–Kisin prism (A, I) with $\mathrm{Spf}(A/I) \xrightarrow{\simeq} X$, see for instance [DLMS22, Example 3.4] or [GR22, proof of Theorem 5.10]. The statement now follows from property (2) of product with any other prism in X_{Δ}. ☐

This allows us to put a t-structure on prismatic (F-)crystals in perfect complexes on X smooth over $\mathcal{O}_K$, as follows.

Construction 3.8. Let us assume first that X is separated and smooth over $\mathcal{O}_K$. By Proposition 3.7, we see that one can find a family of Breuil–Kisin prisms (A_λ, I_λ) indexed by $\lambda \in \Lambda$ which jointly covers X_{Δ}. Now we may form the Cech nerve of the cover $\bigsqcup_{\lambda \in \Lambda}(A_\lambda, I_\lambda) \twoheadrightarrow *$, whose n-th spot is given by $(A_{\lambda_1}, I_{\lambda_1}) \times \cdots \times (A_{\lambda_{n+1}}, I_{\lambda_{n+1}})$ indexed by Λ^{n+1}. Below we will simply denote this Cech nerve by $\mathrm{Spf}(A^{[\bullet]})$ with the corresponding family of rings $A^{[\bullet]}$, so each $A^{[n]}$ really stands for the above family of rings indexed by Λ^{n+1}. By (p, I)-completely faithfully flat descent (c.f. [Mat22, Theorem 5.8] and [BS23, Theorem 2.2]), the ∞-category $(\mathrm{F}\text{-})\mathrm{Crys}^{\mathrm{perf}}(X)$ is given by the cosimplicial limit of the derived ∞-categories of (F-)perfect complexes on $\mathrm{Spf}(A^{[\bullet]})$, we define a t-structure by requiring an (F-)crystals in perfect complexes lives in ≤ 0 part (resp. ≥ 0 part) if the underlying complexes on $\mathrm{Spf}(A^{[\bullet]})$ is concentrated in cohomological degrees ≤ 0 (resp. ≥ 0).

By Proposition 3.7 (2), the maps from $A^{[0]}$ to any of $A^{[\bullet]}$ are all (p, I)-completely flat, so are all the maps composed with the Frobenius on $A^{[\bullet]}$. Hence all these maps are already flat as $A^{[0]}$ consists of (p, I)-complete Noetherian rings. Therefore it suffices to check the cohomological degrees for the underlying complexes on $\mathrm{Spf}(A_\lambda)$'s. We call this the *standard t-structure*, below we shall check that it is indeed a t-structure and is independent of the choice of the family (A_λ, I_λ). We refer readers to [BBD82, Chapter 1] and [Lur17, Section 1.2.1] for a general discussion on t-structures.

Lemma 3.9. *Construction 3.8 defines a t-structure on* $(\mathrm{F}\text{-})\mathrm{Crys}^{\mathrm{perf}}(X)$*. When X is quasi-compact, this t-structure is exhaustive and separated.*

Proof. Given $\mathcal{F}$ (resp. $\mathcal{G}$) in $(\mathrm{F}\text{-})\mathrm{Crys}^{\mathrm{perf}}(X)$ with its values on $\mathrm{Spf}(A_\lambda)$'s concentrated in degrees ≤ 0 (resp. ≥ 1), we see immediately that $\mathrm{Map}_{(\mathrm{F}\text{-})\mathrm{Crys}^{\mathrm{perf}}(X)}(\mathcal{F}, \mathcal{G})$ is the cosimplicial limit of contractible spaces:

This is simply because their values on each $\mathrm{Spf}(A^\bullet)$'s have contractible mapping spaces for degree reason. Therefore we get that the mapping space above is itself a contractible space. To see the existence of $\tau^{\leq 0}$ truncation, one just notices that term-wise $\tau^{\leq 0}$ truncation still satisfies the crystal condition thanks to the discussion prior to this Lemma. Moreover, the resulting object is a perfect complex: By crystal condition it suffices to check this on the family of rings $A^{[0]} := \{A_\lambda\}_{\lambda\in\Lambda}$, but this is automatic as Breuil–Kisin prisms are regular by Corollary 3.6. Due to flatness of $A^{[0]} \to A^{[i]} \xrightarrow{\varphi_{A^{[i]}}} A^{[i]}$, we see that the ≤ 0 truncation of $\varphi^*_{A^{[i]}}\mathcal{F}(A^{[i]})$ is the same as the Frobenius twist of the ≤ 0 truncation of $\mathcal{F}(A^{[i]})$. Hence the linearized Frobenius extends canonically to the truncations, proving the existence of truncations for objects in $\text{F-Crys}^{\mathrm{perf}}(X)$.

When X is quasi-compact, one can find a disjoint union of finitely many Breuil–Kisin prisms to cover $X_{\mathbb{\Delta}}$, and the exhaustiveness and separatedness follows from the same properties of the standard t-structure on $\mathrm{Perf}(R)$ for any regular ring R. □

Granting the claim that in the separated case the standard t-structure is independent of the choice of covering families of Breuil–Kisin prisms, we make the following definition.

Definition 3.10. Let X be smooth over $\mathcal{O}_K$, we define the *standard t-structure* on $\text{(F-)Crys}^{\mathrm{perf}}(X)$ by: An object lives in ≤ 0 part (resp. ≥ 0 part) if its restriction to $\text{(F-)Crys}^{\mathrm{perf}}(U)$ is so for any separated open $U \subset X$. We say an object in $\text{(F-)Crys}^{\mathrm{perf}}(X)$ is *coherent* if it belongs to the heart of the standard t-structure, we denote this heart by $\text{(F-)Crys}^{\mathrm{coh}}(X)$. By taking its associated homotopy category, we may regard $\text{(F-)Crys}^{\mathrm{coh}}(X)$ as an abelian category.

Applying the usual Zariski descent to Breuil–Kisin prisms, one sees that it suffices to check the above condition for a family of separated opens $U \subset X$ which jointly covers X. What is left to show is the claim about independence of the choice of covering family of Breuil–Kisin prisms. In fact, we prove the following stronger statement.

Proposition 3.11. *Let X be separated and smooth over $\mathcal{O}_K$. Fix a covering family of Breuil–Kisin prisms $\bigsqcup_{\lambda\in\Lambda}(A_\lambda, I_\lambda) \twoheadrightarrow *$ which gives rise to a t-structure as in Construction 3.8. Let $(A', I') \in X_{\mathbb{\Delta}}$ and suppose that $\mathrm{Spf}(\overline{A'}) \to X$ is p-completely flat, then the evaluation functor $\mathrm{Crys}^{\mathrm{perf}}(X) \to \mathrm{Perf}(A')$ is t-exact with respect to the standard t-structures on both sides.*

In particular, being in the heart of the t-structure with respect to one covering family of Breuil–Kisin prisms forces its value on any other Breuil–Kisin prism to concentrate in degree 0. This way, we see that the t-structure we get is indeed independent of the choice of covering families of Breuil–Kisin prisms.

Proof. Let (C_λ, K_λ) be the product of (A_λ, I_λ) and (A', I') as in Proposition 3.7. Then property (2) in said Proposition implies that $\mathrm{Spf}(C_\lambda)$ gives rise to a (p, I')-completely flat cover of $\mathrm{Spf}(A')$. By definition, the product $(C^\bullet, K^\bullet)$ of $(A^\bullet, I^\bullet)$ with (A', I') exists and is given by the Cech nerve of the cover $\mathrm{Spf}(C^{[0]}) := \bigsqcup_{\lambda\in\Lambda}\mathrm{Spf}(C_\lambda) \twoheadrightarrow \mathrm{Spf}(A')$.

Now let $\mathcal{F} \in \mathrm{Crys}^{\mathrm{coh}}(X)$ be in the heart of the t-structure defined by (A_λ, I_λ). Then its values on $C^{[\bullet]}$ defines a descent datum giving rise to the value on A' (here we suppresses the ideal for simplicity). By Proposition 3.7 (3), we know that all of $C^{[\bullet]}$ are (p, I)-completely flat over $A^{[0]}$. Since $A^{[0]}$ consists of derived (p, I)-complete rings which are Noetherian, we see that all of $C^{[\bullet]}$ are flat over $A^{[0]}$. The complexes in descent datum are (non-canonically) given by $\mathcal{F}(C^{[\bullet]}) \simeq \mathcal{F}(A^{[0]}) \otimes_{A^{[0]}} C^{[\bullet]}$, we learn that they consist of modules sitting in degree 0. Now Lemma 3.12 below ensures that the descent $\mathcal{F}(A')$ also lives in degree 0, hence proving the assertion. □

Lemma 3.12. *Let $A' \to C^{[0]}$ be a J-completely faithfully flat map of derived J-complete rings, where $J = (f_1, \cdots, f_r) \subset A'$ is a finitely generated ideal. Denote the Cech nerve by $C^{[\bullet]}$. Then in the equivalence $D_{J-comp}(A') \simeq \lim_\Delta D_{J-comp}(C^{[\bullet]})$ given by J-complete faithfully flat descent, a descent datum of the right hand side consisting of complexes $M^{[\bullet]}$ concentrated in cohomological degree 0 has its descent $M \in D_{J-comp}(A')$ concentrated in cohomological degree 0 as well.*

Proof. Since the descent M is calculated by $\lim_\Delta M^{[\bullet]}$, it is concentrated in cohomological degrees ≥ 0. Below we show the reverse cohomological degree estimate.

We look at the following base change formula for any natural number m:

$$M \otimes^L_{A'} \mathrm{Kos}(A'; f_i^m) \otimes^L_{\mathrm{Kos}(A'; f_i^m)} \mathrm{Kos}(C^{[0]}; f_i^m) \cong M^{[0]} \otimes^L_{C^{[0]}} \mathrm{Kos}(C^{[0]}; f_i^m).$$

The right hand side lives in degrees ≤ 0 by our condition that $M^{[0]}$ lives in degree 0. Therefore we see $M \otimes^L_{A'} \mathrm{Kos}(A'; f_i^m)$ also lives in degrees ≤ 0, as $A' \to C^{[0]}$ is J-completely faithfully flat. By the same argument, the transition maps $\pi_0(M \otimes^L_{A'} \mathrm{Kos}(A'; f_i^{m+1})) \to \pi_0(M \otimes^L_{A'} \mathrm{Kos}(A'; f_i^m))$ are all surjective. Using the fact that M is derived J-complete, we get that $M = \mathrm{R}\lim_m \left(M \otimes^L_{A'} \mathrm{Kos}(A'; f_i^m)\right)$, which lives in degrees ≤ 0 due to the above analysis. $\square$

Let us point out that we do not know if the converse to the above holds true: Namely given a J-complete A-module M, base change M along a J-completely flat ring map $A \to B$ might not be concentrated in degree 0 anymore. Now we can show the aforementioned independence of the choice of Breuil–Kisin prisms.

Corollary 3.13. *Let X be a smooth formal scheme over $\mathcal{O}_K$. An object $\mathcal{F} \in (\mathrm{F}\text{-})\mathrm{Crys}^{\mathrm{perf}}(X)$ lives in ≤ 0 part (resp. ≥ 0 part) if and only if any of the following equivalent condition holds:*

1. *There exists a covering family of Breuil–Kisin prisms (A_λ, I_λ) in X_{Δ}, such that $\mathcal{F}(A_\lambda, I_\lambda)$ lives in $D^{\leq 0}(A_\lambda)$ (resp. $D^{\geq 0}(A_\lambda)$).*
2. *For any Breuil–Kisin prism $(A, I) \in X_{\Delta}$, we have $\mathcal{F}(A, I) \in D^{\leq 0}(A)$ (resp. $D^{\geq 0}(A)$).*
3. *There exists a covering family of prisms $(A, I) \in X_{\Delta}$ with $\mathrm{Spf}(\overline{A}) \to X$ being p-completely flat, such that $\mathcal{F}(A, I) \in D^{\leq 0}(A)$ (resp. $D^{\geq 0}(A)$).*
4. *For any prism $(A, I) \in X_{\Delta}$ such that $\mathrm{Spf}(\overline{A}) \to X$ is p-completely flat, we have $\mathcal{F}(A, I) \in D^{\leq 0}(A)$ (resp. $D^{\geq 0}(A)$).*

Proof. We shall prove the "living in ≤ 0 part" statement, as the other part can be proved similarly. It is easy to see that (4) implies (3) and (2), (2) implies (1), and (1) implies that $\mathcal{F}$ lives in ≤ 0 part. The last condition implies (4): This is exactly Proposition 3.11.

What is left to show is that (3) also implies that $\mathcal{F}$ lives in ≤ 0 part. Now we look at the truncation $\mathcal{G} := \tau^{>0}(\mathcal{F})$. Equivalently, we need to show $\mathcal{G}$ is zero. By Proposition 3.11 and condition (3), we see that $\mathcal{G}$ has value 0 on a covering family of prisms, hence it is zero. $\square$

Definition 3.14 (I-torsionfree crystal)**.** We say an $\mathcal{E} \in (\mathrm{F}\text{-})\mathrm{Crys}^{\mathrm{coh}}(X)$ is I-torsionfree if the map of coherent crystals: $\mathcal{I}_{\Delta} \otimes_{\mathcal{O}_{\Delta}} \mathcal{E} \to \mathcal{E}$ is injective, i.e. has no kernel with respect to the standard t-structure. Equivalently, this is asking $\mathcal{E}(A)$ to be I-torsionfree for all (or a covering family of) Breuil–Kisin prisms $(A, I) \in X_{\Delta}$.

Remark 3.15. We have the following fully faithful inclusion of categories:

$$(\mathrm{F}\text{-})\mathrm{Crys}^{\mathrm{vect}}(X) \subset (\mathrm{F}\text{-})\mathrm{Crys}^{\mathrm{an}}(X) \subset (\mathrm{F}\text{-})\mathrm{Crys}^{I\text{-tf}} \subset (\mathrm{F}\text{-})\mathrm{Crys}^{\mathrm{coh}}(X),$$

for the last inclusion, where the target is regarded as an abelian category (since it is the heart of the standard t-structure), see [GR22, Theorem 5.10]. By definition, we have a natural fully faithful inclusion of ∞-categories

$$(\mathrm{F}\text{-})\mathrm{Crys}^{\mathrm{coh}}(X) \subset (\mathrm{F}\text{-})\mathrm{Crys}^{\mathrm{perf}}(X).$$

Recall in [BS23, §3] the authors introduced the notion of Laurent F-crystals (see Definition 3.2 in loc. cit.). According to the Corollary 3.7 in loc. cit., Laurent F-crystals on a bounded p-adic formal scheme X are functorially identified with the locally constant derived category of the adic generic fiber of X:

$$D_{\mathrm{perf}}(X_{\Delta}, \mathcal{O}_{\Delta}[1/I_{\Delta}]^\wedge_p)^{\varphi=1} \simeq D^{(b)}_{lisse}(X_\eta, \mathbb{Z}_p).$$

Notice that the right hand side has its own standard t-structure.

Lemma 3.16. *Let X be a smooth formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$, then the base change functor compose with the above étale realization functor $\left(- \otimes_{\mathcal{O}_{\Delta}} \mathcal{O}_{\Delta}[1/I_{\Delta}]^\wedge_p\right)^{\varphi=1} \colon \mathrm{F}\text{-}\mathrm{Crys}^{\mathrm{perf}}(X) \to D^{(b)}_{lisse}(X_\eta, \mathbb{Z}_p)$ is t-exact.*

Proof. It suffices to prove the statement locally on X, so we may assume $X = \mathrm{Spf}(R)$ is affine which has a Breuil–Kisin prism (A, I) with $A/I = R$ and the Frobenius $\varphi_A \colon A/I[1/p] \to A/(\varphi(I))[1/p]$ is finite étale: Indeed, we may assume that X admits a toric chart, then the Breuil–Kisin prism constructed in

[DLMS22, Example 3.4] has these properties. Suppose $\mathcal{E} \in \text{F-Crys}^{\text{coh}}(X)$, we need to show its étale realization $T(\mathcal{E}) \in D^{(b)}_{lisse}(X_\eta, \mathbb{Z}_p)$ sits in cohomological degree 0.

To that end, let $R_\infty := A_{\text{perf}}/I$ be the reduction of the perfection of (A, I), whose adic generic fiber is an affinoid perfectoid pro-finite-étale cover of X. It suffices to show that the restriction of $T(\mathcal{E})$ to $\text{Spf}(R_\infty)_\eta$ sits in cohomological degree 0. Tracing through the proof of [BS23, Corollary 3.7], we see that the restriction of $T(\mathcal{E})$ is computed (via the tilting equivalence) by the φ-invariants (as a pro-étale sheaf) of $\mathcal{E}(A_{\text{perf}})[1/I]^\wedge_p = \mathcal{E}(A) \otimes_A A_{\text{perf}}[1/I]^\wedge_p$. By definition, the perfect complex $\mathcal{E}(A)$ is given by a finitely presented A-module in degree 0. The ring map $A \to A_{\text{perf}}[1/I]^\wedge_p$ is p-completely flat, hence is flat thanks to Noetherianity of A. Therefore we see that $\mathcal{E}(A_{\text{perf}})[1/I]^\wedge_p$ is also concentrated in degree 0. Now our claim is equivalent to the statement that $\varphi - 1$ on $\mathcal{E}(A_{\text{perf}})[1/I]^\wedge_p$ is surjective (as pro-étale $\mathbb{Z}_p$-sheaves on $A_{\text{perf}}[1/I]^\wedge_p$). This is well-known, but let us still sketch a proof: Since $\mathcal{E}(A_{\text{perf}})[1/I]^\wedge_p$ is derived p-complete, the cokernel is also derived p-complete, therefore it suffices to show its mod p reduction is 0. Equivalently it suffices to show the (derived) φ-invariants of the derived mod p reduction $\mathcal{E}(A_{\text{perf}})[1/I]/p$ lives in degrees ≤ 0, let us denote it by $T(\mathcal{E}/^L p)$.

Finally we may use [BS23, Proposition 3.6]: The authors show that after trivializing $T(\mathcal{E}/^L p)$ on an étale neighborhood U of $\text{Spec}(A_{\text{perf}}[1/I]/p)$, so that $T(\mathcal{E}/^L p)(U)$ is just an $\mathbb{F}_p$ complex (which we need to show to be concentrated in degrees ≤ 0), we have an isomorphism $T(\mathcal{E}/^L p)(U) \otimes_{\mathbb{F}_p} \mathcal{O}_U \simeq \mathcal{E}(A_{\text{perf}})[1/I]/^L p \otimes_{A_{\text{perf}}[1/I]/p} \mathcal{O}_U$, as pro-étale sheaves. Now we claim victory as the right hand side lives in degrees ≤ 0. ☐

Another notion of central interest to our article is the *height* of an I-torsionfree prismatic F-crystal.

Definition 3.17. Let X be a smooth formal scheme over $\text{Spf}(\mathcal{O}_K)$, and let $(\mathcal{E}, \varphi_{\mathcal{E}}) \in \text{F-Crys}^{I\text{-tf}}(X)$. We say it has *height* in $[a, b]$ if the linearized Frobenius

$$\varphi_{\mathcal{E}} \colon \varphi^*_{\mathcal{O}_{\Delta}} \mathcal{E} \longrightarrow \mathcal{E}[1/I]$$

has its image squeezed in the following manner: $I^b\mathcal{E} \subset \text{Im}(\varphi_{\mathcal{E}}) \subset I^a\mathcal{E}$, viewed as sub-$\mathcal{O}_{\Delta}$-modules in $\mathcal{E}[1/I]$. When $[a, b] = [0, h]$, we simply say it is effective of height $\leq h$, this terminology is compatible with [DLMS22, Definition 3.14].

Remark 3.18. One can check height of $(\mathcal{E}, \varphi_{\mathcal{E}})$ on a covering family of prisms whose reduction is flat over X.

Example 3.19. Fix notation as above, the prismatic structure sheaf with its Frobenius is effective and has height ≤ 0. Assuming that X is smooth proper of relative equidimension d over $\text{Spf}(\mathcal{O}_K)$, then the i-th derived pushforward of $\mathcal{O}_{\Delta}$ has its I-torsionfree quotient being effective of height $\leq \min\{i, d\}$: [BS22, Corollary 15.5] shows the $\leq i$ inequality, and the $\leq d$ inequality can be directly proved using [LL20, Lemma 7.8]. In fact the bound in terms of dimension d holds more generally, one can translate an argument due to Bhatt (see [GR22, Remark 8.11]) to the following concrete estimate: if $X \to Y$ is an equi-d-dimensional smooth (without properness!) map of smooth formal schemes over $\text{Spf}(\mathcal{O}_K)$ and let $(\mathcal{E}, \varphi_{\mathcal{E}})$ be an I-torsionfree effective F-crystal on X with height $\leq h$, then for any integer i the i-th derived pushforward $(R^i f_{\Delta,*}\mathcal{E}, \varphi_{\mathcal{E}})$ has its I-torsionfree quotient of height $\leq (d + h)$. Our paper aims at generalizing these height estimates, see Corollary 5.5.

3.2. **Prismatic F-gauges.** In this subsection we shall recall the notion of prismatic F-gauges, introduced by Drinfeld and Bhatt–Lurie. Let us start with F-gauges over a quasiregular semiperfectoid ring.

Definition 3.20 ([Bha22, Definition 5.5.17 and Example 6.1.7])**.** Let S be a quasiregular semiperfectoid ring.

A *prismatic gauge* on $\text{Spf}(S)$ is a (p, I)-complete filtered complex $\text{Fil}^\bullet E$ over $\text{Fil}^\bullet_N \Delta_S$. A *prismatic F-gauge* $E = (E, \text{Fil}^\bullet E, \widetilde{\varphi}_E)$ on $\text{Spf}(S)$ consists of the following data:

- a prismatic gauge $(E, \text{Fil}^\bullet E)$, called *the underlying gauge* of E;
- a map of filtered complexes

$$\widetilde{\varphi}_E : \text{Fil}^\bullet E \longrightarrow I^{\mathbb{Z}} \Delta_S \otimes_{\Delta_S} E,$$

which is linear over the filtered map $\varphi_{\Delta_S} \colon \text{Fil}^\bullet_N \Delta_S \to I^\bullet \Delta_S$;

- such that the filtered linearization
$$\varphi_E : \mathrm{Fil}^\bullet E \otimes_{\mathrm{Fil}^\bullet_N \Delta_S} I^{\mathbb{Z}}\Delta_S \longrightarrow I^{\mathbb{Z}} E$$
is a filtered isomorphism in $\mathcal{DF}_{(p,I)\text{-comp}}(I^{\mathbb{Z}}\Delta_S) \simeq \mathrm{D}_{(p,I)\text{-comp}}(\Delta_S)$.

Following the convention of [Bha22], we call $\mathrm{Fil}^\bullet E$ a *Nygaardian filtration* on E. So a prismatic gauge is nothing but a prismatic crystal equipped with a Nygaardian filtration.

Definition 3.21. Let S be a quasiregular semiperfectoid ring.

(1) We use (F-)Gauge(Spf(S)) to denote the category of prismatic (F-)gauges over Spf(S).
(2) We use (F-)Gauge*(Spf(S)) with $* \in \{\mathrm{vect}, \mathrm{perf}\}$ to denote the subcategory of (F-)Gauge(Spf(S)) consisting of those $E \in$ (F-)Gauge(Spf(S)) such that the underlying filtered $\mathrm{Fil}^\bullet_N \Delta_S$-complex $\mathrm{Fil}^\bullet E$ is filtered (p, I)-completely finite projective (resp. filtered (p, I)-completely perfect), see Definition 2.10 and Definition 2.19.
(3) We use (F-)Gauge$^{\mathrm{coh}}$(Spf(S)) to denote the full subcategory of (F-)Gauge$^{\mathrm{perf}}$(Spf(S)) such that the underlying gauge is a module over Δ_S filtered by submodules.

Remark 3.22. The mapping space between two coherent (F-)gauges on Spf(S) is discrete.

Below let us show that for gauges over quasiregular semiperfectoid rings, there is no difference between filtered completely finite projective (resp. filtered completely perfect) versus filtered finite projective (resp. filtered perfect).

Proposition 3.23. *Let S be a quasiregular semiperfectoid ring and let $\mathrm{Fil}^\bullet E \in \mathrm{Gauge}(\mathrm{Spf}(S))$. Then we have:*

(1) $\mathrm{Fil}^\bullet E$ is filtered finite projective over $\mathrm{Fil}^\bullet_N \Delta_S$ if and only if it is filtered (p, I)-completely finite projective over $\mathrm{Fil}^\bullet_N \Delta_S$; and
(2) $\mathrm{Fil}^\bullet E$ is filtered perfect over $\mathrm{Fil}^\bullet_N \Delta_S$ if and only if it is filtered (p, I)-completely perfect over $\mathrm{Fil}^\bullet_N \Delta_S$.

Proof. According to [ALB23, Lemma 4.28], the pair $(\Delta_S, \mathrm{Fil}^1_N \Delta_S)$ is henselian. Then by [ALB23, Lemma 4.27] (or [Sta23, Tag 0D4A]), we see that every finite projective $(\mathrm{Gr}^0_N \Delta_S = S)$-module can be lifted to a finite projective Δ_S-module. Moreover, we claim that for every finite projective Δ_S-module P, every direct summand $\overline{P_1} \subset \overline{P} := P \otimes_{\Delta_S} S$ can be lifted to a direct summand $P_1 \subset P$: By the previous sentence, we may first lift $\overline{P}_1$ to a finite projective P_1 over Δ_S, then (by the finite projectiveness of P_1) we may lift the map $\overline{P_1} \subset \overline{P}$ to a map $P_1 \to P$, which is necessarily also a direct summand because $\mathrm{Fil}^1_N \Delta_S$ is contained in the Jacobson ideal of Δ_S. Our proposition now follows from Proposition 2.35 and Proposition 2.30 respectively. $\square$

Remark 3.24. For a map of quasiregular semiperfectoid rings $S_1 \to S_2$ in X_{qrsp} and $* \in \{\emptyset, \mathrm{vect}, \mathrm{perf}\}$, the completed filtered base change induces a natural functor
$$\Phi_{(S_1,S_2)} : (\text{F-})\mathrm{Gauge}^*(\mathrm{Spf}(S_1)) \longrightarrow (\text{F-})\mathrm{Gauge}^*(\mathrm{Spf}(S_2)),$$
$$(E, \mathrm{Fil}^\bullet E, (\widetilde{\varphi}_E)) \longmapsto (E \bigotimes_{\Delta_{S_1}} \Delta_{S_2}, \mathrm{Fil}^\bullet E \bigotimes_{\mathrm{Fil}^\bullet_N \Delta_{S_1}} \mathrm{Fil}^\bullet_N \Delta_{S_2}, (\widetilde{\varphi}_E \otimes \varphi_{\Delta_{S_2}})).$$
From the construction, the induced functor on underlying (F-)crystals is the natural pullback functor.

By taking limit of the above categories, we obtain the definition of (F-)gauges for more general schemes.

Definition 3.25. Let X be a quasi-syntomic formal scheme. The category of *prismatic (F-)gauges* over X is defined as the limit of ∞-categories
$$(\text{F-})\mathrm{Gauge}^*(X) := \lim_{S \in X_{\mathrm{qrsp}}} (\text{F-})\mathrm{Gauge}^*(\mathrm{Spf}(S)),$$
where $* \in \{\emptyset, \mathrm{vect}, \mathrm{perf}, \mathrm{coh}\}$.

Remark 3.26. There is a natural functor from (F-)gauges to (F-)crystals by forgetting the filtration, and it is compatible with the limit formula in [BS23, Proposition 2.14].

Remark 3.27. In the stacky formalism of [Bha22], the categories (F-)Gauge*(Spf(S)) for $* \in \{\emptyset, \mathrm{vect}, \mathrm{perf}\}$ are equivalent to the categories of complexes (resp. vector bundles, resp. perfect complexes) over the filtered prismatization $X^{\mathcal{N}}$ (resp. syntomification X^{syn}) of X. However the analogous statement is probably not true when $* = \mathrm{coh}$: Our notion of coherent F-gauges should be stronger than being a "coherent sheaf on the syntomification of X".

A key fact is that the category of (F-)gauges form a sheaf under quasi-syntomic topology. Before proving this, let us record some useful flatness criteria in commutative algebra.

Lemma 3.28 ([Sta23, Tag 0H7N])**.** *Let $A \to B$ be a ring map, let $f \in A$ be a nonzerodivisor in both A and B. Assume that $A/f \to B/f$ and $A[1/f] \to B[1/f]$ are (faithfully) flat, then $A \to B$ is (faithfully) flat. If $J \subset A$ is a finitely generated ideal, then the analogous statement with all "(faithfully) flat" replaced by "J-completely (faithfully) flat" also holds.*

Proof. Let us prove the "flat" statement first, the faithful part is easy. Let M be an A module, denote by N the submodule consisting of f-power torsion elements, then M/N is an f-torsion free A module. Since $\mathrm{Tor}_i^A(M, B)$ is squeezed between $\mathrm{Tor}_i^A(N, B)$ and $\mathrm{Tor}_i^A(M/N, B)$, it is equivalent to showing the later two vanish for $i > 0$.

Let us treat N first. Since N is the filtered colimit of its finitely generated submodules N' each of which is annihilated by a power (depending on the finitely generated submodule) of f, we are further reduced to showing: If f^c annihilates N', then $N' \otimes_A^L B$ lives in degree 0. Now we write $N' \otimes_A^L B = N' \otimes_{A/f^c}^L A/f^c \otimes_A^L B = N' \otimes_{A/f^c}^L B/f^c$: For the second equality we use the fact that f is a nonzerodivisor in B. We may appeal to [Sta23, Tag 051C] and see that $A/f^c \to B/f^c$ is flat.

Now we treat M/N. First we observe $M/N \otimes_A^L B \otimes_B^L B/f = (M/N)/^L f \otimes_{A/f}^L B/f$ which lives in degree 0 by f-torsion free assumption on M/N and flatness of $A/f \to B/f$. Therefore we see that $\mathrm{Tor}_i^A(M/N \otimes_A^L B)$ are f-invertible for all $i > 0$: Indeed one simply observes that $\mathrm{Tor}_i^A(M/N \otimes_A^L B)/f$ and $\mathrm{Tor}_{i-1}^A(M/N \otimes_A^L B)[f]$ are subquotients of $\mathrm{H}_i((M/N \otimes_A^L B) \otimes_B^L B/f)$. Hence we may invert f and get $\mathrm{Tor}_i^A(M/N, B) = \mathrm{Tor}_i^{A[1/f]}(M/N[1/f], B[1/f]) = 0$ for all $i > 0$, here the first equality follows from previous sentence and the second equality follows from the flatness assumption on $A[1/f] \to B[1/f]$.

For the J-complete analog, one simply runs the above argument with the starting assumption that M is an A/J module. $\square$

Lemma 3.29. *Let $(A, \mathrm{Fil}_\bullet(A)) \to (B, \mathrm{Fil}_\bullet(B))$ be a filtered map of commutative unital rings equipped with increasing exhaustive multiplicative $\mathbb{N}$-indexed filtrations. If the graded ring map $\mathrm{Gr}_\bullet(A) \to \mathrm{Gr}_\bullet(B)$ is (faithfully) flat, then $A \to B$ is (faithfully) flat. If $J \subset \mathrm{Fil}_0(A)$ is a finitely generated ideal and if the increasing filtrations on A and B are J-completely exhaustive, then the analogous statement with all "(faithfully) flat" replaced by "J-completely (faithfully) flat" also holds.*

Proof. It is equivalent to showing the following: let $K \subset A$ be an ideal, then $K \otimes_A B \to B$ is injective (for the faithfully flat statement we need to further show that this map is not surjective unless $K = A$ is the unit ideal). Now we equip K with the induced filtration $\mathrm{Fil}_i(K) = K \cap \mathrm{Fil}_i(A)$, and consider the following filtered map $\mathrm{Fil}_\bullet(K \otimes_A B) \coloneqq \mathrm{Fil}_\bullet(K) \otimes_{\mathrm{Fil}_\bullet(A)}^L \mathrm{Fil}_\bullet(B) \to \mathrm{Fil}_\bullet(B)$. The source is equipped with exhaustive increasing $\mathbb{N}$-indexed filtration and the above map has its graded pieces given by $\mathrm{Gr}_\bullet(K \otimes_A B) = \mathrm{Gr}_\bullet(K) \otimes_{\mathrm{Gr}_\bullet(A)}^L \mathrm{Gr}_\bullet(B) \to \mathrm{Gr}_\bullet(B)$. As $\mathrm{Gr}_\bullet K$ is a graded ideal inside of $\mathrm{Gr}_\bullet A$, the (faithfully) flatness of $\mathrm{Gr}_\bullet(A) \to \mathrm{Gr}_\bullet(B)$ implies that the above has its source living in degree 0 and the induced map is injective (and not surjective unless $\mathrm{Gr}_\bullet(K) = \mathrm{Gr}_\bullet(A)$). This together with the snake lemma implies that $K \otimes_A B \to B = \mathrm{colim}_{i \to \infty}(\mathrm{Fil}_i(K \otimes_A B) \to \mathrm{Fil}_i(B))$ is injective (and not surjective unless $K = A$). For the completely (faithfully) flat statement, one just runs the above argument with the assumption that K contains J. $\square$

The quasi-syntomic descent of (F-)gauges follows from [Bha22, Theorem 5.5.10 and Remark 5.5.18], and since our definition (which does not involve the stacky approach) a priori does not agree with the one given in Bhatt's notes, we give a direct proof.

Proposition 3.30 (c.f. [Bha22, Theorem 5.5.10 and Remark 5.5.18])**.** *Let $S \to S^{(0)}$ be a quasi-syntomic cover of quasiregular semiperfectoid rings, and let $S^{(\bullet)}$ be the p-completed Čech nerve. Then for $* \in \{\emptyset, \mathrm{perf}, \mathrm{vect}\}$, we have a natural equivalence*

$$(\text{F-})\mathrm{Gauge}^*(\mathrm{Spf}(S)) \simeq \lim_{[n]\in\Delta} (\text{F-})\mathrm{Gauge}^*(\mathrm{Spf}(S^{(n)})).$$

Proof. Let us treat the descent of gauges first, we fix a map $R \to S$ where R is perfectoid with its associated perfect prism $(A, I = (d))$. Via Rees's construction (see [Bha22, §2.2]), we can regard gauges as graded (p, I)-complete complexes over the graded ring $\mathrm{Rees}(\mathrm{Fil}^\bullet_N) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i_N t^{-i}$. By Proposition 2.38, the descent of $*$-gauges follows from the following two statements:

(1) The map $\mathrm{Rees}(\mathrm{Fil}^\bullet_N(\Delta_S)) \to \mathrm{Rees}(\mathrm{Fil}^\bullet_N(\Delta_{S^{(0)}}))$ is (p, I)-completely faithfully flat.
(2) For any map of qrsp algebras $S \to \widetilde{S}$ with $\widetilde{S}^{(0)} := \widetilde{S}\widehat{\otimes}_S S^{(0)}$, we have a filtered isomorphism:

$$\mathrm{Fil}^\bullet_N(\Delta_{\widetilde{S}})\widehat{\otimes}_{\mathrm{Fil}^\bullet_N(\Delta_S)}\mathrm{Fil}^\bullet_N(\Delta_{S^{(0)}}) \xrightarrow{\cong} \mathrm{Fil}^\bullet_N(\Delta_{\widetilde{S}^{(0)}}).$$

Let us prove (1) first. Using Lemma 3.28 (with the f there being t here), we see that it suffices to show that both the underlying ring map $\Delta_S \to \Delta_{S^{(0)}}$ and the graded algebra map

$$\bigoplus_{i\in\mathbb{N}} \mathrm{Gr}^i_N(\Delta_S)t^{-i} \to \bigoplus_{i\in\mathbb{N}} \mathrm{Gr}^i_N(\Delta_{S^{(0)}})t^{-i}$$

are completely faithfully flat. By [BS22, Theorem 12.2], the graded algebra is identified with Rees's construction of the (twisted) conjugate filtered Hodge–Tate algebra (in particular the filtration satisfies the assumption in Lemma 3.29), namely

$$\bigoplus_{i\in\mathbb{N}} \mathrm{Gr}^i_N(\Delta_S)t^{-i} \cong \bigoplus_{i\in\mathbb{N}} \mathrm{Fil}^{\mathrm{conj}}_i \overline{\Delta}_S\{i\}t^{-i}$$

(similarly with S replaced by $S^{(0)}$). Applying again Lemma 3.28 (with the f there being d/t here), we see that it suffices to show $\overline{\Delta}_S \to \overline{\Delta}_{S^{(0)}}$ and $\bigoplus_{i\in\mathbb{N}} \mathrm{Gr}^{\mathrm{conj}}_i(\overline{\Delta}_S)t^{-i} \to \bigoplus_{i\in\mathbb{N}} \mathrm{Gr}^{\mathrm{conj}}_i(\overline{\Delta}_{S^{(0)}})t^{-i}$ are completely faithfully flat. Here we have also used the fact that $\Delta_S \to \Delta_{S^{(0)}}$ is (p, I)-completely faithfully flat if and only if $\overline{\Delta}_S \to \overline{\Delta}_{S^{(0)}}$ is p-completely faithfully flat. Now we apply Lemma 3.29 to the conjugate filtered Hodge–Tate rings, we see that all we need to show is the p-completely faithfully flatness of the graded algebra map (of conjugate filtration). Since the graded algebra of conjugate filtration is the (derived) Hodge algebra, the map (p-completed) is identified with $\Gamma^*_S(\mathbb{L}_{S/R}[-1])^\wedge \to \Gamma^*_{S^{(0)}}(\mathbb{L}_{S^{(0)}/R}[-1])^\wedge$. Since the target is $S^{(0)}$-algebra, we see this map can be factored through the base change along $S \to S^{(0)}$ which is p-completely faithfully flat. Finally, we are reduced to showing the map $\Gamma^*_{S^{(0)}}(\mathbb{L}_{S/R}[-1]\widehat{\otimes}_S S^{(0)})^\wedge \to \Gamma^*_{S^{(0)}}(\mathbb{L}_{S^{(0)}/R}[-1])^\wedge$ (induced by the natural map of cotangent complexes) is p-completely faithfully flat. This follows from the fact that

$$\mathrm{Cone}(\mathbb{L}_{S/R}[-1]\widehat{\otimes}_S S^{(0)} \to \mathbb{L}_{S^{(0)}/R}[-1]) = \mathbb{L}_{S^{(0)}/S}[-1]$$

is a p-completely flat $S^{(0)}$-module thanks to the quasi-syntomicity of $S \to S^{(0)}$ and the assumption that $S^{(0)}$ is semiperfectoid.

Let us now show the statement (2) above. Such a filtered map is a filtered isomorphism if and only if its underlying and graded maps are both isomorphisms. By [BS22, Theorem 12.2], the graded map is identified with:

$$\Big(\bigoplus_{i\in\mathbb{N}} \mathrm{Fil}^{\mathrm{conj}}_i(\overline{\Delta}_{\widetilde{S}})t^{-i}\Big)\widehat{\otimes}_{\left(\bigoplus_{i\in\mathbb{N}} \mathrm{Fil}^{\mathrm{conj}}_i(\overline{\Delta}_S)t^{-i}\right)}\Big(\bigoplus_{i\in\mathbb{N}} \mathrm{Fil}^{\mathrm{conj}}_i(\overline{\Delta}_{S^{(0)}})t^{-i}\Big) \to \Big(\bigoplus_{i\in\mathbb{N}} \mathrm{Fil}^{\mathrm{conj}}_i(\overline{\Delta}_{\widetilde{S}^{(0)}})t^{-i}\Big).$$

Via the Rees's construction, the above being an isomorphism is equivalent to the following filtered map being an isomorphism:

$$\mathrm{Fil}^{\mathrm{conj}}_\bullet(\overline{\Delta}_{\widetilde{S}})\widehat{\otimes}_{\mathrm{Fil}^{\mathrm{conj}}_\bullet(\overline{\Delta}_S)}\mathrm{Fil}^{\mathrm{conj}}_\bullet(\overline{\Delta}_{S^{(0)}}) \xrightarrow{\cong} \mathrm{Fil}^{\mathrm{conj}}_\bullet(\overline{\Delta}_{\widetilde{S}^{(0)}}).$$

Now for a map of increasingly $\mathbb{N}$-indexed p-completely exhaustively filtered complexes, being a filtered isomorphism is equivalent to its graded map being an isomorphism. After taking graded algebras, we are reduced to showing the following graded map being an isomorphism:

$$\Gamma^*_{\widetilde{S}}(\mathbb{L}_{\widetilde{S}/R}[-1])^\wedge\widehat{\otimes}_{\Gamma^*_S(\mathbb{L}_{S/R}[-1])^\wedge}\Gamma^*_{S^{(0)}}(\mathbb{L}_{S^{(0)}/R}[-1])^\wedge \xrightarrow{\cong} \Gamma^*_{\widetilde{S}^{(0)}}(\mathbb{L}_{\widetilde{S}^{(0)}/R}[-1])^\wedge.$$

The left hand side is identified with $\Gamma^*_{\widetilde{S}^{(0)}}\big(\mathrm{Cone}(\mathbb{L}_{S/R}[-1]\widehat{\otimes}_S\widetilde{S}^{(0)} \to \mathbb{L}_{S^{(0)}/R}[-1]\widehat{\otimes}_{S^{(0)}}\widetilde{S}^{(0)} \oplus \mathbb{L}_{\widetilde{S}/R}[-1]\widehat{\otimes}_{\widetilde{S}}\widetilde{S}^{(0)})\big)^\wedge$, and we are further reduced to showing the following map of $\widetilde{S}^{(0)}$-modules is an isomorphism:

$$\mathrm{Cone}(\mathbb{L}_{S/R}[-1]\widehat{\otimes}_S\widetilde{S}^{(0)} \to \mathbb{L}_{S^{(0)}/R}[-1]\widehat{\otimes}_{S^{(0)}}\widetilde{S}^{(0)} \oplus \mathbb{L}_{\widetilde{S}/R}[-1]\widehat{\otimes}_{\widetilde{S}}\widetilde{S}^{(0)}) \to \mathbb{L}_{\widetilde{S}^{(0)}/R}[-1]^\wedge,$$

which follows from the fact that $\widetilde{S}^{(0)}$ is given by p-completely (derived) tensor of $S^{(0)}$ and $\widetilde{S}$ over S and functoriality of cotangent complexes. This finishes the proof of the graded part of the Nygaard-filtered base change formula, and it also simultaneously proves the part about underlying complexes: Since in the process we have showed the following filtered map being an isomorphism:

$$\mathrm{Fil}^{\mathrm{conj}}_\bullet(\overline{\Delta}_{\widetilde{S}})\widehat{\otimes}_{\mathrm{Fil}^{\mathrm{conj}}_\bullet(\overline{\Delta}_S)}\mathrm{Fil}^{\mathrm{conj}}_\bullet(\overline{\Delta}_{S^{(0)}}) \xrightarrow{\cong} \mathrm{Fil}^{\mathrm{conj}}_\bullet(\overline{\Delta}_{\widetilde{S}^{(0)}}).$$

Taking its underlying map, we get the following isomorphism: $\overline{\Delta}_{\widetilde{S}}\widehat{\otimes}_{\overline{\Delta}_S}\overline{\Delta}_{S^{(0)}} \xrightarrow{\cong} \overline{\Delta}_{\widetilde{S}^{(0)}}$, which in turn shows that its I-completely deformed map is also an isomorphism: $\Delta_{\widetilde{S}}\widehat{\otimes}_{\Delta_S}\Delta_{S^{(0)}} \xrightarrow{\cong} \Delta_{\widetilde{S}^{(0)}}$ (by derived Nakayama's Lemma).

As for the descent of F-gauges, since we have established the descent of gauges we are reduced to showing descent of the Frobenius morphism which follows from flat descent. □

Lemma 3.31. *In Proposition 3.30, the descent of coherent (F-)gauges is again coherent.*

Proof. Perfectness was already shown in loc. cit., so all we need to check is the cohomological concentration property, which is equivalent to both $\mathrm{Fil}^\bullet$ and $\mathrm{Gr}^\bullet$ being concentrated in cohomological degree 0. Both can be seen by exactly following the proof of Lemma 3.12. □

The following is our main theorem in this subsection, which says that any coherent I-torsionfree prismatic F-crystal can be canonically extended to a coherent prismatic F-gauge equipped with "the saturated Nygaardian filtration". This extends the special case of $X = \mathrm{Spf}(\mathcal{O}_K)$ in [Bha22, Theorem 6.6.13] to arbitrary smooth p-adic formal schemes over $\mathrm{Spf}(\mathcal{O}_K)$.

Theorem 3.32. *Let X be a smooth formal scheme over $\mathcal{O}_K$.*

(1) There is a functor

$$\Pi_X : \text{F-Crys}^{I\text{-tf}}(X) \longrightarrow \text{F-Gauge}^{\mathrm{coh}}(X),$$

characterized by the requirement that for any $S \in X_{\mathrm{qrsp}}$, where $\mathrm{Spf}(S) \to X$ is p-completely flat, the restriction of $\Pi_X(\mathcal{E})$ on $\mathrm{Spf}(S)_\Delta$ satisfies $\mathrm{Fil}^\bullet(\Pi_X(\mathcal{E})(\Delta_S)) = \widetilde{\varphi}_{\mathcal{E}}^{-1}(I^\bullet\mathcal{E}(\Delta_S))$. Here we are regarding both the source and target as additive categories.

(2) The functor Π_X is right adjoint to the forgetful functor from I-torsionfree F-gauges to I-torsionfree F-crystals. In fact, for any pair of $(E_1, \mathrm{Fil}^\bullet E_1, \widetilde{\varphi}_{E_1}) \in \text{F-Gauge}^{\mathrm{coh}}(X)$ and $(\mathcal{E}_2, \widetilde{\varphi}_{\mathcal{E}_2}) \in \text{F-Crys}^{I\text{-tf}}(X)$, we have an identification of homomorphisms:

$$\mathrm{Hom}_{\text{F-Crys}^{\mathrm{coh}}(X)}(E_1, \mathcal{E}_2) = \mathrm{Hom}_{\text{F-Gauge}^{\mathrm{coh}}(X)}(E_1, \Pi_X(\mathcal{E}_2)).$$

(3) The functor Π_X is compatible with étale pullback in X.

We need some preparatory discussion on local situations first. We fix the following assumption for the rest of the subsection.

Situation 3.33. Let $U = \mathrm{Spf}(R)$ be an affine open of X, and let $(A, I = (d))$ be an oriented Breuil–Kisin prism of U whose existence is again guaranteed by deformation theory (see for example [DLMS22, Example 3.4]). Denote the perfection $(A_{\mathrm{perf}}, IA_{\mathrm{perf}})$ by $(A^{(0)}, IA^{(0)})$. We denote the value of $\mathcal{E}$ on (A, I) and $(A^{(0)}, IA^{(0)})$ by M and $M^{(0)}$, they are equipped with linearized Frobenii φ_M and $\varphi_{M^{(0)}}$ respectively.

Construction 3.34. Recall that the F-crystal structure provides us with a linearized Frobenii $\varphi_A^*M \xrightarrow{\varphi_M} M[1/I]$ and $\varphi_{A^{(0)}}^*M^{(0)} \xrightarrow{\varphi_{M^{(0)}}} M^{(0)}[1/I]$. We define the *twisted filtration* on φ_A^*M (resp. $\varphi_{A^{(0)}}^*M^{(0)}$) by the following formula

$$\mathrm{Fil}^\bullet_{\mathrm{tw}}\varphi_A^*M := \varphi_M^{-1}\big(I^\bullet M\big) \text{ (resp. } \mathrm{Fil}^\bullet_{\mathrm{tw}}\varphi_{A^{(0)}}^*M^{(0)} := \varphi_{M^{(0)}}^{-1}\big(I^\bullet M^{(0)}\big)).$$

Lemma 3.35. *Let E be an I-torsionfree F-crystal having height in $[a,b]$, and denote the associated graded of the twisted filtration on $\varphi_A^*M = \varphi_A^*E(A)$ above by N, viewed as a graded $R[u] = \mathrm{Gr}_{I^{\mathbb{N}}}(A)$-module. Then N is u-torsionfree, lives in degrees $\geq a$, and $N^{\deg=i} \xrightarrow{\cdot u} N^{\deg=i+1}$ is an isomorphism provided $i \geq b$. In particular, both N and $N[1/u]/N$ have bounded p-power torsion.*

Proof. Since E is I-torsionfree, we know that the linearized Frobenius is injective. Therefore we have an identification: $\mathrm{Fil}^i_{\mathrm{tw}}(M) = \mathrm{Im}(\varphi_M) \cap I^i M$. Consequently, we have

$$N^{\deg=i} = \mathrm{Im}(\mathrm{Im}(\varphi_M) \cap I^i M \to I^i M/I^{i+1}M),$$

which immediately implies u-torsionfreeness. The rest of the second sentence follows from the height condition. It follows that the $R[u]$-module N is generated by its degree $[a,b]$ pieces, and since each graded piece is finitely generated over R, we see N is finitely generated over $R[u]$. Since $R[u]$ is Noetherian, we see that N has bounded p-power torsion. The R-module $N[1/u]/N$ is given by the direct sum of infinite copies of M/I together with $\mathrm{Coker}(N^{\deg=i} \xrightarrow{\cdot u^{b-i}} N^{\deg=b})$ where $i \in [a, b-1]$. This explicit description shows the boundedness claim of $N[1/u]/N$. □

We get the following consequence concerning perfectness of the twisted filtration on φ_A^*M.

Lemma 3.36. *The filtered $I^{\mathbb{N}}A$-module $\mathrm{Fil}^\bullet_{\mathrm{tw}}\varphi_A^*M$ is filtered perfect.*

Proof. Since A is I-adically complete, we may apply Proposition 2.30, therefore we need to check that

(1) The twisted filtration is eventually constant (see Definition 2.29);
(2) The underlying φ_A^*M is perfect over A; and
(3) the filtered base change $\mathrm{Fil}^\bullet_{\mathrm{tw}}\varphi_A^*M \otimes_{I^{\mathbb{N}}A} A/I$ is filtered perfect over A/I.

By Lemma 2.32.(1) and Lemma 3.35, the twisted filtration is constant after a-th filtration. (2) follows from the fact that M is A-perfect.

Below let us show (3) above. Using the resolution $0 \to I \otimes_A I^{\mathbb{N}}A\langle -1\rangle \to I^{\mathbb{N}}A \to A/I \to 0$ of the filtered $I^{\mathbb{N}}A$-algebra $A/I = \mathrm{Gr}^0(I^{\mathbb{N}}A)$, we see that the filtered base change has its filtered pieces given by:

$$\mathrm{Fil}^i\big(\mathrm{Fil}^\bullet_{\mathrm{tw}}\varphi_A^*M \otimes_{I^{\mathbb{N}}A} A/I\big) = \mathrm{Coker}(I \otimes_A \mathrm{Fil}^{i-1}_{\mathrm{tw}}\varphi_A^*M \to \mathrm{Fil}^i_{\mathrm{tw}}\varphi_A^*M).$$

Therefore we see that $\mathrm{Fil}^{\geq(b+1)}$ of the filtered base change is 0. Note that by our convention, the Rees's construction $\mathrm{Rees}(A/I) = R[t]$ is graded by $-\mathbb{N}$, and the filtered base change has bounded above grading by the previous sentence. By Corollary 2.26, we are reduced to showing

$$\mathrm{Rees}(\mathrm{Fil}^\bullet_{\mathrm{tw}}\varphi_A^*M \otimes_{I^{\mathbb{N}}A} A/I) \otimes_{\mathrm{Rees}(A/I)} A/I \cong \mathrm{Gr}^*(\mathrm{Fil}^\bullet_{\mathrm{tw}}\varphi_A^*M \otimes_{I^{\mathbb{N}}A} A/I) \cong N \otimes_{R[u]} R = N/^L u$$

is graded perfect over R. Here the first equivalence follows from the fact that $A/I = \mathrm{Cone}(\mathrm{Rees}(A/I) \xrightarrow{\cdot t} \mathrm{Rees}(A/I)$, and the second equivalence follows from the compatibility in Lemma 2.31. By Lemma 3.35, we see that $N/^L u$ is concentrated in gradings between a and b. By Noetherianity of R, we see that in each degree, they are all finitely generated R-modules, therefore they are all perfect R-complexes as R is regular. □

We need the following auxiliary lemma which says that the saturated Nygaardian filtration on $\Pi_U(E)\mid_{A^{(0)}}$ has a model over the filtered ring $(A, I^{\mathbb{N}})$.

Lemma 3.37. *We have the following filtered base change formula between twisted filtrations constructed in Construction 3.34:*

$$\mathrm{Fil}^\bullet_{\mathrm{tw}}\varphi_A^*M\widehat{\otimes}_{(A,I^{\mathbb{N}})}(A^{(0)}, I^{\mathbb{N}}A^{(0)}) \xrightarrow{\cong} \mathrm{Fil}^\bullet_{\mathrm{tw}}\varphi_{A^{(0)}}^*M^{(0)},$$

as well as the following filtered base change formula between the twisted filtration and the saturated Nygaardian filtration:

$$\mathrm{Fil}^\bullet_{\mathrm{tw}}\varphi_{A^{(0)}}^*M^{(0)}\widehat{\otimes}_{(A^{(0)},I^{\mathbb{N}}A^{(0)}),\varphi^{-1}}(A^{(0)}, \varphi^{-1}(I^{\mathbb{N}}A^{(0)}) = \mathrm{Fil}^\bullet_N(A^{(0)})) \xrightarrow{\cong} \mathrm{Fil}^\bullet M^{(0)}.$$

Here we are using the fact that $A^{(0)}$ is a perfect prism, hence its Frobenius is invertible.

Proof. Note that we have the following relation between Rees algebras associated with various filtered rings showing up in the statement:

$$\mathrm{Rees}(I^{\mathbb{N}}A)\widehat{\otimes}_A A^{(0)} \cong \mathrm{Rees}(I^{\mathbb{N}}A^{(0)})^{\wedge} \text{ and}$$
$$\mathrm{Rees}(I^{\mathbb{N}}A^{(0)})\widehat{\otimes}_{A^{(0)},\varphi^{-1}} A^{(0)} \cong \mathrm{Rees}(\varphi^{-1}(I^{\mathbb{N}}A^{(0)}))^{\wedge} = \mathrm{Rees}(\mathrm{Fil}^{\bullet}_N {\mathbb{\Delta}}_{A^{(0)}/I})^{\wedge}.$$

Via Rees's construction (see [Bha22, §2.2]), our first base change formula is equivalent to the following formula:

$$\mathrm{Fil}^{\bullet}_{\mathrm{tw}}\varphi_A^* M\widehat{\otimes}_A A^{(0)} \xrightarrow{\cong} \mathrm{Fil}^{\bullet}_{\mathrm{tw}}\varphi^*_{A^{(0)}} M^{(0)}$$

which follows from the fact that $A \to A^{(0)}$ is flat and under this flat map the φ_M is base changed to $\varphi_{M^{(0)}}$. Similarly, our second base change formula is equivalent to:

$$\mathrm{Fil}^{\bullet}_{\mathrm{tw}}\varphi^*_{A^{(0)}} M^{(0)}\widehat{\otimes}_{A^{(0)},\varphi^{-1}} A^{(0)} \xrightarrow{\cong} \mathrm{Fil}^{\bullet} M^{(0)}.$$

Recall that the filtration $\mathrm{Fil}^{\bullet} M^{(0)}$ is defined by the preimage of I-adic filtration under the semi-linear Frobenius $\widetilde{\varphi}_{M^{(0)}}$. Therefore our second base change formula follows from the fact that $\varphi^{-1}_{A^{(0)}}$ is an isomorphism and upon base changing the source via $\varphi^{-1}_{A^{(0)}}$ the linearized Frobenius $\varphi_{M^{(0)}}$ becomes the semi-linear Frobenius $\widetilde{\varphi}_{M^{(0)}}$. □

The following crucial proposition, which is inspired by the proof of [Bha22, Lemma 6.6.10], shows that the saturated Nygaardian filtration satisfies filtered base change formula in a particular situation.

Proposition 3.38. *Let $R^{(0)} := A^{(0)}/I$ and let S be a p-completely flat quasi-syntomic $R^{(0)}$-algebra, then the natural filtered base change is a filtered isomorphism:*

$$\left(M^{(0)}, \widetilde{\varphi}_{\mathcal{E}}^{-1}(I^{\bullet}M^{(0)})\right)\widehat{\otimes}_{(A^{(0)},\mathrm{Fil}^{\bullet}_N)}({\mathbb{\Delta}}_S, \mathrm{Fil}^{\bullet}_N) \xrightarrow{\cong} \left(\mathcal{E}({\mathbb{\Delta}}_S), \widetilde{\varphi}_{\mathcal{E}}^{-1}(I^{\bullet}\mathcal{E}({\mathbb{\Delta}}_S))\right).$$

Consequently, if $S_1 \to S_2$ is a morphism in X_{qrsp} and suppose they are p-completely flat over some $R^{(0)}$ constructed out of an oriented Breuil–Kisin prism $(A, I = (d))$, then the natural filtered base change is a filtered isomorphism:

$$\left(\mathcal{E}({\mathbb{\Delta}}_{S_1}), \widetilde{\varphi}_{\mathcal{E}}^{-1}(I^{\bullet}\mathcal{E}({\mathbb{\Delta}}_{S_1}))\right)\widehat{\otimes}_{({\mathbb{\Delta}}_{S_1},\mathrm{Fil}^{\bullet}_N)}({\mathbb{\Delta}}_{S_2}, \mathrm{Fil}^{\bullet}_N) \xrightarrow{\cong} \left(\mathcal{E}({\mathbb{\Delta}}_{S_2}), \widetilde{\varphi}_{\mathcal{E}}^{-1}(I^{\bullet}\mathcal{E}({\mathbb{\Delta}}_{S_2}))\right).$$

Proof. Since the map always induces an isomorphism of the underlying complex, thanks to $\mathcal{E}$ being a crystal, all we need to check is that the tensor product filtration (defined by the left hand side of the above equation) and the saturated Nygaardian filtration (defined by the right hand side of the above equation) agree.

Now we recall in the proof of [Bha22, Lemma 6.6.10], Bhatt observes the following characterization of the saturated Nygaardian filtration:

Porism 3.39 (Follows from the proof of [Bha22, Lemma 6.6.10])**.** *Let A be a ring and let $(M_1, \mathrm{Fil}^{\bullet} M_1) \xrightarrow{\widetilde{\varphi}} (M_2, \mathrm{Fil}^{\bullet} M_2)$ be a map of two decreasing filtered objects in $\mathcal{DF}(A)$ such that*

(i) $\mathrm{Fil}^{\ll 0} M_1 \to M_1$ *is an isomorphism, all* $\mathrm{Fil}^{\bullet} M_1$*'s are connective;*
(ii) The graded complexes $\mathrm{Gr}^{\bullet}(M_1)$ *are coconnective;*
(iii) $(\mathrm{Fil}^{\bullet} M_2)$ *are A-modules with injective transitions; and*
(iv) The graded map $\mathrm{Gr}(\widetilde{\varphi})\colon \mathrm{Gr}^{\bullet}(M_1) \to \mathrm{Gr}^{\bullet}(M_2)$ *have coconnective cone.*

Then $(M_1, \mathrm{Fil}^{\bullet} M_1)$ is also an honest decreasingly filtered A-module with its filtration given by $\mathrm{Fil}^{\bullet} M_1 = \widetilde{\varphi}^{-1}(\mathrm{Fil}^{\bullet} M_2)$.

Proof. This is an exercise in homological algebra, below we give some hints. The conditions (i) and (ii) imply that $(M_1, \mathrm{Fil}^{\bullet} M_1)$ is an honest decreasingly filtered A-module. Then condition (iii) and (iv) implies that $\mathrm{Fil}^{i+1} M_1 = \mathrm{Fil}^i M_1 \cap \widetilde{\varphi}^{-1}(\mathrm{Fil}^{i+1} M_2)$, finishing the proof. □

We shall verify that the filtered map

$$\left(M^{(0)}, \widetilde{\varphi}_{\mathcal{E}}^{-1}(I^{\bullet}M^{(0)})\right)\widehat{\otimes}_{(A^{(0)},\mathrm{Fil}^{\bullet}_N)}({\mathbb{\Delta}}_S, \mathrm{Fil}^{\bullet}_N) \xrightarrow{\widetilde{\varphi}_{\mathcal{E}}} \left(\mathcal{E}({\mathbb{\Delta}}_S)[1/I], I^{\bullet}\mathcal{E}({\mathbb{\Delta}}_S)\right).$$

satisfies the conditions of Porism 3.39, which will show that the tensor product filtration agrees with the saturated Nygaardian filtration. The condition (i) is stable under filtered base change between honestly filtered algebras, therefore the tensor product filtration satisfies the condition (i) as $(M^{(0)}, \widetilde{\varphi}_{\mathcal{E}}^{-1}(I^\bullet M^{(0)}))$ clearly satisfies it. The condition (iii) is easily seen to be satisfied. Finally we need to verify (ii) and (iv) in the Porism above concerning the behavior of graded pieces.

For (ii): The graded pieces of the tensor product filtration is given by $\mathrm{Gr}(M^{(0)})\widehat{\otimes}_{\mathrm{Gr}_N(A^{(0)})}\mathrm{Gr}_N(\mathbb{\Delta}_S)$. By Lemma 3.37, we get the following description of $\mathrm{Gr}(M^{(0)}) = \mathrm{Gr}_{\mathrm{tw}}(\varphi_A^* M)\widehat{\otimes}_{R[u]}\mathrm{Gr}_N(A^{(0)})$. Here the tensor is p-completed and the base change map is given by taking graded map of composition of the following filtered maps (where u is the image of d in I/I^2):

$$(A, I^{\mathbb{N}}A) \to (A^{(0)}, I^{\mathbb{N}}A^{(0)}) \xrightarrow{\varphi_{A^{(0)}}^{-1}} (A^{(0)}, \mathrm{Fil}_N^\bullet).$$

In the proof of Proposition 3.30, we have seen that the map $\mathrm{Gr}_N(A^{(0)}) \to \mathrm{Gr}_N(\mathbb{\Delta}_S)$ is p-completely flat. Precomposing with the p-completely flat map $R[u] \to \mathrm{Gr}_N(A^{(0)})$, we see that the induced map $R[u] \to \mathrm{Gr}_N(\mathbb{\Delta}_S)$ is p-completely flat. Therefore we see that $\mathrm{Gr}_{\mathrm{tensor}}(E(\mathbb{\Delta}_S)) = \mathrm{Gr}_{\mathrm{tw}}(\varphi_A^* M)\widehat{\otimes}_{R[u]}\mathrm{Gr}_N(\mathbb{\Delta}_S)$ is concentrated in degree 0 because $\mathrm{Gr}_{\mathrm{tw}}(\varphi_A^* M)$ has bounded p-power torsion (Lemma 3.35).

As for (iv): just like the proof of [Bha22, Lemma 6.6.10], we may identify the map

$$\mathrm{Gr}_{\mathrm{tensor}}(E(\mathbb{\Delta}_S)) \xrightarrow{\widetilde{\varphi}_{\mathcal{E}}} \mathrm{Gr}_{I\text{-adic}}(E(\mathbb{\Delta}_S)[1/I])$$

as the p-completely base change of $\mathrm{Gr}_N(M^{(0)}) \to \mathrm{Gr}_N(M^{(0)})[1/u]$ along the map $\mathrm{Gr}_N(A^{(0)}) \to \mathrm{Gr}_N(\mathbb{\Delta}_S)$. Using Lemma 3.37 again, we may identify the above map further as the following p-completely base changed map

$$(\mathrm{Gr}_{\mathrm{tw}}(\varphi_A^* M) \to \mathrm{Gr}_{\mathrm{tw}}(\varphi_A^* M)[1/u])\,\widehat{\otimes}_{R[u]}\mathrm{Gr}_N(\mathbb{\Delta}_S).$$

Therefore it suffices to notice that $(\mathrm{Gr}_{\mathrm{tw}}(\varphi_A^* M)[1/u]/\mathrm{Gr}_{\mathrm{tw}}(\varphi_A^* M))$ has bounded p-power torsion (Lemma 3.35). $\square$

Now we are ready to prove Theorem 3.32, let us stress again that it is inspired by [Bha22, Lemma 6.6.10].

Proof of Theorem 3.32. (1): we adopt the notation in the discussion right after the statement. Let $U = \mathrm{Spf}(R)$ be an affine open of X. Let $R^{(0)} := A^{(0)}/I$ and let $R^{(\bullet)}$ be the Cech nerve of the quasi-syntomic cover $R \to R^{(0)}$ with their absolute prismatic cohomology $A^{(\bullet)} := \mathbb{\Delta}_{R^{(\bullet)}}$. We denote $E(A^{(\bullet)})$ by $M^{(\bullet)}$, and by abuse of notation we denote the semi-linear Frobenii on these $M^{(\bullet)}$ by the same symbol $\widetilde{\varphi}_{\mathcal{E}}$.

Since F-gauges form a quasi-syntomic sheaf (Proposition 3.30), we need to first check that the "saturated Nygaardian filtrations" on $\mathcal{E}(\mathbb{\Delta}_{R^{(i)}})$ satisfies filtered base change with respect to the various maps induced by the simplicial maps between the $R^{(i)}$'s. To that end, let $[i] \to [j]$ be an arrow in Δ, we need to verify the natural map

$$\left(M^{(i)}, \widetilde{\varphi}_{\mathcal{E}}^{-1}(I^\bullet M^{(i)})\right)\widehat{\otimes}_{(A^{(i)},\mathrm{Fil}_N^\bullet)}(A^{(j)}, \mathrm{Fil}_N^\bullet) \to \left(M^{(j)}, \widetilde{\varphi}_{\mathcal{E}}^{-1}(I^\bullet M^{(j)})\right)$$

is a filtered isomorphism. This immediately follows from Proposition 3.38.

This shows that the saturated Nygaardian filtration defines an F-gauge $\Pi_U(\mathcal{E}|_U)$ on U, and the construction is easily seen to be functorial in $\mathcal{E}$ and U. The perfectness follows from combining Lemma 3.36 and Lemma 3.37. Next we verify that these $\Pi_U(\mathcal{E}|_U)$'s glue to an F-gauge on X. To that end, it suffices to check the following

Claim 3.40. *For any* $\mathrm{Spf}(S) \in U_{\mathrm{qrsp}}$ *with p-completely flat structural map to U, the value* $\Pi_U(\mathcal{E}|_U)(\mathrm{Spf}(S))$ *has its filtration given by the saturated Nygaardian filtration.*

Granting the above claim, then for any two affine opens U and V, the restriction of $\Pi_U(\mathcal{E}|_U)|_{U\cap V}$ and $\Pi_V(\mathcal{E}|_V)|_{U\cap V}$ will be canonically identified: their values on any $\mathrm{Spf}(S)$ which are flat and quasi-syntomic over $U \cap V$ are canonically identified (the filtrations are given by the saturated Nygaardian filtration), finally we just notice that these $\mathrm{Spf}(S)$'s form a basis of $(U \cap V)_{\mathrm{qrsp}}$, and F-gauges form a quasi-syntomic sheaf (Proposition 3.30).

Proof of the above claim: Let us base change the Cech nerve $R^{(\bullet)}$ along the map $R \to S$, and denote the resulting Cech nerve $S^{(\bullet)}$. The underlying gauge of $\Pi_U(\mathcal{E}|_U)(\mathrm{Spf}(S))$ is the descent of the gauges on $\mathrm{Spf}(S^{(\bullet)})$, and the latter gauges are given by the filtered base change $\left(M^{(i)}, \widetilde{\varphi}_{\mathcal{E}}^{-1}(I^\bullet M^{(i)})\right)\widehat{\otimes}_{(A^{(i)},\mathrm{Fil}_N^\bullet)}(\mathbb{\Delta}_{S^{(i)}}, \mathrm{Fil}_N^\bullet)$. Using Proposition 3.38, we see that the latter gauges are nothing but the saturated Nygaardian filtered module $\mathcal{E}(\mathbb{\Delta}_{S^{(i)}})$. By Lemma 3.31, we see the descent gauge $(\mathcal{E}(\mathbb{\Delta}_S), \mathrm{Fil}_{\mathrm{descent}})$ is coherent, namely it is an honest decreasingly filtered $\mathbb{\Delta}_S$-module. Finally we apply Porism 3.39 for the Frobenius filtered map $(\mathcal{E}(\mathbb{\Delta}_S), \mathrm{Fil}^\bullet_{\mathrm{descent}}) \xrightarrow{\widetilde{\varphi}_\mathcal{E}} (\mathcal{E}(\mathbb{\Delta}_S)[1/I], I^\bullet\mathcal{E}(\mathbb{\Delta}_S))$: we have verified conditions (i) and (ii); the condition (iii) follows from coherence of $\mathcal{E}$ and Corollary 3.13; finally the condition (iv) is stable under descent as taking limit preserves co-connectivity, hence it is satisfied for our map (as this map is the descent of maps which satisfy condition (iv)). Therefore we may conclude that the descent filtration is again the saturated Nygaardian filtration, which finishes our proof of the claim. □

To finish the proof of (1), we need to see that for any $\mathrm{Spf}(S) \in X_{\mathrm{qrsp}}$ with p-completely flat structural map to X, the value $\Pi_X(\mathcal{E})(\mathrm{Spf}(S))$ has its filtration given by the saturated Nygaardian filtration. We may choose a Zariski cover of $\mathrm{Spf}(S)$ by affine opens such that each affine open maps to an affine open in X. Then the filtration on $\Pi_X(\mathcal{E})(\mathrm{Spf}(S))$ is given by descent of saturated Nygaardian filtration (by the above claim), therefore we can use the same argument as above again to conclude that the filtration we get is the saturated Nygaardian filtration on $\mathcal{E}(\mathbb{\Delta}_S)$.

Next we show (2): Let E_1 and $\mathcal{E}_2$ be as in the statement. Notice that mapping spaces $\mathrm{Map}_{\text{F-Crys}^{\mathrm{coh}}(X)}(E_1, \mathcal{E}_2)$ and $\mathrm{Map}_{\text{F-Gauge}^{\mathrm{coh}}(X)}(E_1, \Pi_X(\mathcal{E}_2))$ are discrete, we see that the Hom groups are glued from the Hom groups on affine opens of $U \subset X$, so we have reduced ourselves to the case of X being an affine. In this case, the Hom groups are computed by the corresponding Hom groups of values of E_1, $\mathcal{E}_2$, and $\Pi_X(\mathcal{E}_2)$ at those $\mathrm{Spf}(S) \in X_{\mathrm{qrsp}}$ which are flat over X. So we are finally reduced to checking the analogous statement for X replaced by these qrsp $\mathrm{Spf}(S)$. Since any Frobenius-equivariant map $E_1(\mathbb{\Delta}_S) \to \mathcal{E}_2(\mathbb{\Delta}_S)$ uniquely extends to a filtered map $E_1(\mathbb{\Delta}_S) \to \Pi_X(\mathcal{E}_2)(\mathbb{\Delta}_S)$, as the latter is equipped with the saturated Nygaardian filtration, the two Hom groups are canonically identified as desired.

Lastly we deal with (3): Let $V \to U \to X$ be two affines in $X_{\acute{e}t}$, we just need to show the natural functor $\text{F-Gauge}^{\mathrm{perf}}(U) \to \text{F-Gauge}^{\mathrm{perf}}(V)$ sends $\Pi_U(\mathcal{E}\mid_U) \mapsto \Pi_V(\mathcal{E}\mid_V)$. Choose a Breuil–Kisin prism (A, I) for U, the étale map $V \to U$ canonically lifts (A, I) to a Breuil–Kisin prism (A', IA') over (A, I) in $U_{\mathbb{\Delta}}$. Notice that $A \to A'$ is (p, I)-completely flat, hence flat by Noetherianity of A. Tracing through the argument of proving (1), we see that it suffices to show the twisted filtrations on $\varphi_A^*\mathcal{E}(A)$ and $\varphi_{A'}^*\mathcal{E}(A')$ are related via

$$\mathrm{Fil}_{\mathrm{tw}}\varphi_A^*\mathcal{E}(A)\widehat{\otimes}_A A' \xrightarrow{\cong} \mathrm{Fil}_{\mathrm{tw}}\varphi_{A'}^*\mathcal{E}(A'),$$

which follows directly from the flatness of $A \to A'$ and how the twisted filtrations are defined. □

Remark 3.41. In fact we may define the functor on all of coherent F-crystals, by pulling back filtrations from their I-torsionfree quotients. The resulting graded modules will agree with the graded modules of the F-gauges of their I-torsionfree quotients. Since we do not need this greater generality, we restrict ourselves to working with I-torsionfree coherent F-crystals only.

3.3. **Weight filtration on graded pieces of gauges.** In this subsection, we introduce a so-called weight filtration on the associated graded of a gauge.

To start, we first define the notion of *weight* for an (F-)gauge, using a reduction functor that sends the graded piece of a gauge over X to a graded complex over $\mathcal{O}_X$.

Construction 3.42 (c.f. Construction 2.23)**.** Let X be a quasi-syntomic p-adic formal scheme, and let $* \in \{\emptyset, \mathrm{perf}, \mathrm{vect}\}$. We define the *reduction functor* on the category $\mathcal{DG}^*_{p\text{-comp}}(X_{\mathrm{qrsp}}, \mathrm{Gr}^\bullet_N\mathbb{\Delta}) \simeq \lim_{S\in X_{\mathrm{qrsp}}} \mathcal{DG}^*_{p\text{-comp}}(\mathrm{Gr}^\bullet_N\mathbb{\Delta}_S)$ as the following composition of functors

$$\mathrm{Red}_X := \mathcal{DG}^*_{p\text{-comp}}(\mathrm{Gr}^\bullet_N\mathbb{\Delta}) \xrightarrow{-\otimes_{\mathrm{Gr}^\bullet_N\mathbb{\Delta}}\mathcal{O}_{\mathrm{qrsp}}} \mathcal{DG}^*_{p\text{-comp}}(\mathcal{O}_X);$$

$$M^\bullet \longmapsto M^\bullet \bigotimes_{\mathrm{Gr}^\bullet_N\mathbb{\Delta}} \mathcal{O}_{\mathrm{qrsp}},$$

where $\mathcal{O}_{\mathrm{qrsp}} = \mathrm{Gr}^0_N\mathbb{\Delta}$ is regarded as a graded $\mathrm{Gr}^\bullet_N\mathbb{\Delta}$-algebra via projection onto degree 0 piece. The i-th graded piece of $\mathrm{Red}_X(-)$ is denoted as $\mathrm{Red}_{i,X}(-)$. Here we implicitly use the p-completely flat descent of p-complete complexes (resp. perfect complexes, resp. vector bundles) for the target category, namely

$$\mathcal{DG}^*_{p\text{-comp}}(\mathcal{O}_X) \simeq \lim_{S\in X_{\mathrm{qrsp}}} \mathcal{DG}^*_{p\text{-comp}}(S).$$

In particular, we have the natural limit formula for the reduction functor

$$\mathrm{Red}_X \simeq \lim_{S \in X_{\mathrm{qrsp}}} \mathrm{Red}_S.$$

When the choice of the formal scheme X is clear, we omit X in the subscripts and use Red and Red_i to abbreviate Red_X and $\mathrm{Red}_{i,X}$. By a slight abuse of notation, for a gauge $(E, \mathrm{Fil}^\bullet E)$ over X, we also abbreviate the notation $\mathrm{Red}_X(\mathrm{Gr}^\bullet E)$ as $\mathrm{Red}_X(E)$ when there is no confusion.

We then define the notion of weights for a gauge.

Definition 3.43. Let X be a quasi-syntomic formal scheme, and let $[a, b]$ be an interval in $\mathbb{R} \cup \{-\infty, \infty\}$. For a graded complex $M^\bullet \in \mathcal{DG}^*_{p\text{-comp}}(X_{\mathrm{qrsp}}, \mathrm{Gr}^\bullet_N {\Delta})$, we say it has *weights* in $[a, b]$ if $M^n = 0$ for $n \ll 0$ and

$$\mathrm{Red}_i(M^\bullet) = 0, \ \forall i \notin [a, b].$$

For a gauge $E = (E, \mathrm{Fil}^\bullet E)$ on X, we say it has *weights* in $[a, b]$ if its associated graded $M^\bullet = \mathrm{Gr}^\bullet E$ is so.

Remark 3.44. In the stacky language, the reduction functor on the category of gauges can be translated in terms of the pullback functor along the closed immersion $X \times B\mathbb{G}_m \to X^N$ as in [Bha22, Rmk. 5.3.14]. Similarly the "weights" defined above corresponds to the "Hodge–Tate weights" defined in loc. cit.

Using the reduction functor, we now introduce the weight filtration on the associated graded of a gauge.

Theorem 3.45. *Let X be a quasi-syntomic p-adic formal scheme, let $* \in \{\emptyset, \mathrm{perf}, \mathrm{vect}\}$, and let $a \le b$ be two integers. Assume $M^\bullet \in \mathcal{DG}^*_{p\text{-comp}}(X_{\mathrm{qrsp}}, \mathrm{Gr}^\bullet_N {\Delta})$ has weights in $[a, b]$. Then there is a unique increasing filtration $\mathrm{Fil}^{\mathrm{wt}}_i(M^\bullet)$ on $M^\bullet$ satisfying the following axioms:*

(1) The filtration is exhaustive;
(2) the filtration "starts at 0": i.e. $\mathrm{Fil}^{\mathrm{wt}}_{\ll 0}(M^\bullet) = 0$; and
(3) the graded piece $\mathrm{Gr}^{\mathrm{wt}}_i(M^\bullet) \simeq N_i \otimes_{\mathcal{O}_X} \mathrm{Gr}^\bullet_N {\Delta}$ where $N_i \in \mathcal{DG}^*_{p\text{-comp}}(\mathcal{O}_X)$ has grading i.

Moreover, the induced filtration $\mathrm{Fil}^{\mathrm{wt}}_i(\mathrm{Red}(M^\bullet)) := \mathrm{Red}(\mathrm{Fil}^{\mathrm{wt}}_i(M^\bullet))$ is the one induced by grading:

$$\mathrm{Fil}^{\mathrm{wt}}_i(\mathrm{Red}(M^\bullet)) \cong \bigoplus_{j \le i} \mathrm{Red}_j(M^\bullet).$$

In particular, there are natural isomorphisms $N_i \cong \mathrm{Red}(\mathrm{Gr}^{\mathrm{wt}}_i(M^\bullet)) \cong \mathrm{Gr}^{\mathrm{wt}}_i(\mathrm{Red}(M^\bullet)) \cong \mathrm{Red}_i(M^\bullet)$ in $\mathcal{D}^*_{p\text{-comp}}(\mathcal{O}_X)$. Therefore, the filtration is indexed by $i \in [a, b]$.

Note that in the special case when $M^\bullet = \mathrm{Gr}^\bullet E$ where $E = (E, \mathrm{Fil}^\bullet E) \in \mathrm{Gauge}^*(X)$, we get a weight filtration $\mathrm{Fil}^{\mathrm{wt}}_i(\mathrm{Gr}^\bullet E)$ on $\mathrm{Gr}^\bullet E$ with $\mathrm{Gr}^{\mathrm{wt}}_i(\mathrm{Gr}^\bullet E) \simeq \mathrm{Red}_i(E) \otimes_{\mathcal{O}_X} \mathrm{Gr}^\bullet_N {\Delta}$.

Proof. The existence part follows exactly from Proposition 2.25, by getting such a filtration when evaluated on each $S \in X_{\mathrm{qrsp}}$: the outcome automatically satisfies the axioms above, hence they will also satisfy graded base change property. This shows the case of $* = \emptyset$, the case of $* =$ perf and vect follows from Corollary 2.26 and Corollary 2.27 respectively. The uniqueness can be seen by exactly following the argument of Proposition 2.25. □

Remark 3.46. For an integer a and some $b \in \mathbb{Z}_{\ge a} \cup \{\infty\}$, we let $\mathcal{DG}^*_{p\text{-comp}}(X, \mathrm{Gr}^\bullet_N {\Delta})^+_{[a,b]}$ be the full subcategory of $\mathcal{DG}^*_{p\text{-comp}}(X, \mathrm{Gr}^\bullet_N {\Delta})$ whose objects have weights in $[a, b]$. Then the natural full embedding $\mathcal{DG}^*_{p\text{-comp}}(X, \mathrm{Gr}^\bullet_N {\Delta})^+_{[a+1,b]} \to \mathcal{DG}^*_{p\text{-comp}}(X, \mathrm{Gr}^\bullet_N {\Delta})^+_{[a,b]}$, admits a left adjoint given by sending an object $M^\bullet$ to the grade complex $M^\bullet / \mathrm{Fil}^{\mathrm{wt}}_a M^\bullet$.

We have the following concrete understanding of the reduction of graded pieces of F-gauges which come from I-torsionfree F-crystals as in Theorem 3.32.

Theorem 3.47. *Let X be a smooth formal scheme over $\mathcal{O}_K$, and let $(\mathcal{E}, \varphi_{\mathcal{E}}) \in \mathrm{F\text{-}Crys}^{I\text{-tf}}(X)$ having height in $[a, b]$.*

(1) Then the gauge $\Pi_X(\mathcal{E})$ constructed in Theorem 3.32 has the reduction of its graded pieces given by a graded coherent sheaf having gradings in $[a, b]$.

(2) In the setting of Situation 3.33, we have the following concrete description of

$$\mathrm{Red}_U(\mathrm{Gr}^\bullet(\Pi_U(\mathcal{E}|_U))) \cong \mathrm{Gr}^\bullet_{\mathrm{tw}}(\varphi_A^* M)/^L u \otimes_R \mathcal{O}_{\mathrm{qrsp}}.$$

We remind readers that the twisted filtration on $\varphi_A^* M$ was discussed around Construction 3.34, and the symbol u stands for the image of $d \in I$ in $\mathrm{Gr}^\bullet(I^{\mathbb{N}}A)$ (so it has degree 1).

Proof. Let us show (2) first. Tracing through the proof of Theorem 3.32, especially Lemma 3.37 and Proposition 3.38 (notice that the S there forms a basis of the qrsp site of U), we get the following relation between twisted filtration on $\varphi_A^* M$ and saturated filtration on $\Pi_U(\mathcal{E})$:

$$\mathrm{Fil}^\bullet_{\mathrm{tw}}(\varphi_A^* M) \otimes_{(A, I^{\mathbb{N}})} \mathrm{Fil}^\bullet_N {\mathbb{\Delta}} \cong \mathrm{Fil}^\bullet_N \Pi_U(\mathcal{E}).$$

Passing to graded pieces and applying the reduction functor, we get:

$$\mathrm{Red}_U(\mathrm{Gr}^\bullet(\Pi_U(\mathcal{E}|_U))) \cong \mathrm{Gr}^\bullet_{\mathrm{tw}}(\varphi_A^* M) \otimes_{R[u]} \mathrm{Gr}^\bullet_N {\mathbb{\Delta}} \otimes_{\mathrm{Gr}^\bullet_N {\mathbb{\Delta}}} \mathcal{O}_{\mathrm{qrsp}} \cong \mathrm{Gr}^\bullet_{\mathrm{tw}}(\varphi_A^* M) \otimes_{R[u]} R(= R[u]/u) \otimes_R \mathcal{O}_{\mathrm{qrsp}}.$$

To prove (1), we may work Zariski locally on X. Now the statement follows from (2) and the explicit knowledge of $\mathrm{Gr}^\bullet_{\mathrm{tw}}(\varphi_A^* M)$ as a graded-$R[u]$-module, see Lemma 3.35. □

In the remainder of this subsection, we discuss the relation between weights of an F-gauge and heights of its underlying F-crystal. First we observe that the Frobenius twist of an F-gauge is eventually I-adically filtered.

Proposition 3.48. *Let S be a quasiregular semiperfectoid ring, let $E = (E, \mathrm{Fil}^\bullet E, \widetilde{\varphi}_E) \in \mathrm{F\text{-}Gauge}^{\mathrm{perf}}(X)$ be of weight $[a,b]$. Denote by $\mathrm{Fil}^\bullet(\varphi^* E)$ the filtered base change of $\mathrm{Fil}^\bullet E$ along $\varphi_{{\mathbb{\Delta}}_S} : \mathrm{Fil}^\bullet_N {\mathbb{\Delta}}_S \to I^\bullet {\mathbb{\Delta}}_S$.*

- *(i) The Frobenius structure $\widetilde{\varphi}_E$ induces a filtered map $v : \mathrm{Fil}^\bullet(\varphi^* E) \to I^\bullet \otimes E$, which is an isomorphism on $\mathrm{Fil}^{\geq b}(-)$. In particular, the filtered complex $\mathrm{Fil}^{\geq b}(\varphi^* E)$ is isomorphic to the I-adic filtration $I^{\geq b} \otimes E$.*
- *(ii) For $i \in \mathbb{Z}$, the weight filtration induces a finite increasing exhaustive filtration of range $[a,b]$ on the map $\mathrm{Gr}^i(v) : \mathrm{Gr}^i(\varphi^* E) \to \bar{I}^i E$, such that its j-th graded piece of the map is*

$$\begin{cases} \mathrm{Red}_j(E) \otimes_S \bar{I}^{i-j} \overline{{\mathbb{\Delta}}}_S \xrightarrow{\simeq} \mathrm{Red}_j(E) \otimes_S \bar{I}^{i-j} \overline{{\mathbb{\Delta}}}_S, & i \geq j; \\ 0 \longrightarrow \mathrm{Red}_j(E) \otimes_S \bar{I}^{i-j} \overline{{\mathbb{\Delta}}}_S, & i < j. \end{cases}$$

Proof. We first notice that as $\mathrm{Fil}^\bullet E$ is a filtered perfect complex over $\mathrm{Fil}^\bullet_N {\mathbb{\Delta}}_S$, its filtered base change

$$\mathrm{Fil}^\bullet E \bigotimes_{\mathrm{Fil}^\bullet_N {\mathbb{\Delta}}_S, \varphi_{{\mathbb{\Delta}}_S}} I^\bullet {\mathbb{\Delta}}_S$$

is also a filtered perfect complex over $I^\bullet {\mathbb{\Delta}}_S$. Moreover, as the ring ${\mathbb{\Delta}}_S$ is (p, I)-complete, the filtered ring $I^\bullet {\mathbb{\Delta}}_S$ is thus filtered complete. In particular, by Proposition 2.18 and the perfectness, the filtered complex $\mathrm{Fil}^\bullet(\varphi^* E)$ and hence $\mathrm{Fil}^{\geq b}(\varphi^* E)$ is also filtered complete and (p, I)-complete automatically. Notice that by the perfectness again, the ${\mathbb{\Delta}}_S$-complex E is I-adic complete. So to show the filtered map $\mathrm{Fil}^{\geq b}(v)$ is a filtered isomorphism, it suffices to check the associated map of the graded pieces.

We then note by construction that the map v can be factored as the composition

$$\mathrm{Fil}^\bullet E \widehat{\bigotimes_{\mathrm{Fil}^\bullet_N {\mathbb{\Delta}}_S, \varphi_{{\mathbb{\Delta}}_S}}} I^\bullet {\mathbb{\Delta}}_S \xrightarrow{u} \mathrm{Fil}^\bullet E \widehat{\bigotimes_{\mathrm{Fil}^\bullet_N {\mathbb{\Delta}}_S, \varphi_{{\mathbb{\Delta}}_S}}} I^{\mathbb{Z}} {\mathbb{\Delta}}_S \xrightarrow[\simeq]{\varphi_E} I^{\mathbb{Z}} E,$$

where the map φ_E is a filtered isomorphism, and to prove (i) it suffices to show that $\mathrm{Gr}^{\geq b}(u)$ is a graded isomorphism. On the other hand, the weight filtration of $\mathrm{Gr}^\bullet E$ induces a finite increasing exhaustive filtration on the graded map $\mathrm{Gr}^\bullet(u)$ above, whose j-th associated graded is

$$V_j \otimes_S \bar{I}^{\mathbb{N}} \overline{{\mathbb{\Delta}}}_S \longrightarrow V_j \otimes_S \bar{I}^{\mathbb{Z}} \overline{{\mathbb{\Delta}}}_S, \tag{$*$}$$

where $V_j \simeq \mathrm{Red}_j(E)$ is a perfect S-complex of graded degrees $j \in [a,b]$. This finishes the proof of (i), since j is assumed to be $\leq b$. By applying $\mathrm{Gr}^i(-)$ at the graded map $(*)$, we further obtain (ii). □

Remark 3.49. It is clear from the proof that Proposition 3.48 can be extended to more general F-gauges $E = (E, \mathrm{Fil}^\bullet E, \widetilde{\varphi}_E)$: we are only using their weights are in $[a,b]$ and $\mathrm{Fil}^\bullet E$ is filtered complete.

In the following, we let $\text{F-Gauge}^{I\text{-tf}}(X)$ be the category of coherent F-gauges with I-torsionfree underlying F-crystals.

Corollary 3.50. *Assume X is smooth over $\mathcal{O}_K$, and $E = (E, \mathrm{Fil}^\bullet E, \widetilde{\varphi}_E) \in \text{F-Gauge}^{I\text{-tf}}(X)$ is of height $[a, a+1]$. Then $E = \Pi_X(\mathcal{E})$ where $\mathcal{E}$ is the underlying F-crystal of E and Π_X is the functor obtained in Theorem 3.32. In particular, E is saturated.*

Proof. We adopt the notation in Situation 3.33. In particular, we let $U = \mathrm{Spf}(R)$ be an affine open of X, and let (A, I) be a Breuil–Kisin prism of U with perfection $(A^{(0)}, IA^{(0)})$ and $S \coloneqq \overline{A^{(0)}}$.

Next we show that the value of E at $\mathrm{Spf}(S)$ is equipped with the saturated Nygaardian filtration. As E is of height $\geq a$, we have $\mathrm{Fil}^a E(S) = E(S)$. Moreover, by Proposition 3.48.(i), $\mathrm{Fil}^{\geq a+1}(\varphi^*_{\mathbb{\Delta}_S} E(S))$ is isomorphic to the I-adic filtration on $I^{a+1}E(S)$ and thus equals to $\varphi_E^{-1}(I^{\geq a+1}E(S))$. Notice that since $\varphi_{\mathbb{\Delta}_S} : \mathrm{Fil}^\bullet_N \mathbb{\Delta}_S \to I^\bullet \mathbb{\Delta}_S$ is a filtered isomorphism, we can untwist $\varphi_{\mathbb{\Delta}_S}$ to get $\mathrm{Fil}^{\geq a+1} E(S) \simeq \widetilde{\varphi}_E^{-1}(I^{\geq a+1}E(S))$. Hence the value $E(S)$ is saturated.

Now the same proof of Theorem 3.32 (1) shows that the value of E at $\mathrm{Spf}(\overline{A^{(i)}/I})$ are equipped with saturated Nygaardian filtrations for all i. Lastly the same proof of Claim 3.40 shows that for any $\mathrm{Spf}(T) \in X_{\mathrm{qrsp}}$ with p-completely flat structural map to U, the value of E at $\mathrm{Spf}(T)$ has saturated Nygaardian filtration, this finishes the proof. ☐

We also give a relation on the weight of an F-gauge and the height of the underlying F-crystal.

Proposition 3.51. *Let X be a quasi-syntomic p-adic formal scheme, and let $E = (E, \mathrm{Fil}^\bullet E, \widetilde{\varphi}_E) \in \text{F-Gauge}^{I\text{-tf}}(X)$.*

(i) If E is of weight $[a,b]$, then its underlying F-crystal $(\mathcal{E}, \varphi_{\mathcal{E}})$ has height within $[a,b]$ as well.
(ii) Conversely, assume $(\mathcal{E}, \varphi_{\mathcal{E}})$ is of height $[a', b']$ with E being saturated, and there is a quasi-syntomic cover of X by perfectoids. Then E is of weight $[a', b']$.

Note that $\mathrm{Fil}^\bullet E$ is saturated when E is an F-gauge in vector bundle (Proposition 3.52). As a consequence, in this case the notion of weight of E and the height of the underlying F-crystal coincide.

Proof. As both weight and height can be checked locally with respect to quasi-syntomic topology, it suffices to assume $X = \mathrm{Spf}(S)$ is quasiregular semiperfectoid. We first assume that E is of weight $[a,b]$. By construction (cf. Theorem 3.45), we have $\mathrm{Fil}^i E = E$ for $i \leq a$. In particular, since $\mathrm{Fil}^i E \subseteq \widetilde{\varphi}_E^{-1}(I^i E)$, the image $\widetilde{\varphi}_E(E)$ and hence the image of the linearlized map $\varphi_E(E)$ are contained in $I^a E$; namely the F-crystal $(\mathcal{E}, \varphi_{\mathcal{E}})$ has height $\geq a$. Moreover, by Proposition 3.48, since the filtered map v is an isomorphism on $\mathrm{Fil}^{\geq b}(-)$, by forgetting the filtration and looking at the underlying map of modules, we get the image of the linearized map $\varphi_E(E)$ contains the submodule $I^b E$, and thus the associated F-crystal is of height $\leq b$. Here we also note that if $\mathrm{Red}_b(E)$ is nonzero, then by looking at the b-th graded factor of the map $(*)$, for any $j < b$ we have

$$\mathrm{Gr}^b(V_j \otimes_S \overline{I}^{\mathbb{N}} \overline{\mathbb{\Delta}}_S) = 0 \longrightarrow \mathrm{Gr}^b(V_j \otimes_S \overline{I}^{\mathbb{Z}} \overline{\mathbb{\Delta}}_S) = V_j \otimes_S \overline{\mathbb{\Delta}}_S \neq 0.$$

In particular, for any given integer $l < b$, the map $\mathrm{Gr}^l u$ and hence $\mathrm{Fil}^l u$ is not surjective and hence not an isomorphism.

For (ii), we assume the underlying F-crystal of E is of height $[a', b']$, and S is perfectoid. Then the image $\varphi_{\mathcal{E}}(\mathcal{E})$ satisfies the inclusions

$$I^{b'} \mathcal{E} \subseteq \varphi_{\mathcal{E}}(\mathcal{E}) \subseteq I^{a'} \mathcal{E}.$$

So by taking the preimage along the $\mathbb{\Delta}_S$-linear map $\varphi_{\mathcal{E}}$, we see $\varphi_{\mathcal{E}}^{-1}(I^{a'}\mathcal{E}) = \varphi^*_{\mathbb{\Delta}_S} \mathcal{E}$, and the map $\varphi_{\mathcal{E}}$ induces an isomorphism $\varphi_{\mathcal{E}}^{-1}(I^{\geq b'}\mathcal{E}) \simeq I^{\geq b'}\mathcal{E}$. On the other hand, since $\varphi_{\mathbb{\Delta}_S}$ is a filtered isomorphism, by the assumption that E has saturated filtration, we get

$$\mathrm{Fil}^\bullet(\varphi^*_{\mathbb{\Delta}_S} E) = \varphi_{\mathcal{E}}^{-1}(I^\bullet \mathcal{E}).$$

This in particular implies that $\mathrm{Fil}^{a'}E = E$, and E is of weight $\geq a'$. Moreover, by Proposition 3.48.(ii), the map $\varphi_E^{-1}(I^iE) \to I^iE$ is not an isomorphism unless $i \geq \max\{l \mid \mathrm{Red}_l(E) \neq 0\}$, where the number $\max\{l \mid \mathrm{Red}_l(E) \neq 0\}$ is by definition the largest possible weight of E (Definition 3.43). Thus E is of weight $\leq b'$. ☐

3.4. **F-gauges in vector bundles and p-divisible groups.** Recall that in [ALB23] the authors established, for every quasi-syntomic p-adic formal scheme X, an anti-equivalence between p-divisible groups over X and admissible prismatic Dieudonné crystals over X (see [ALB23, Theorem 1.4.4]). Here a prismatic Dieudonné crystal over X is the same as an F-crystal in vector bundles over $\mathrm{Spf}(R)_{\Delta}$ which is effective of height ≤ 1, and admissibility is a technical condition introduced in [ALB23, Def. 4.5], and is automatic if X is regular and admits a quasi-syntomic cover by perfectoids, (see [ALB23, Definition 1.3.1 and Remark 1.4.9][7]). Below, for quasi-syntomic X, we let $\mathrm{F\text{-}Crys}^{\mathrm{adm,vect}}_{[0,1]}(X)$ be the subcategory of $\mathrm{F\text{-}Crys}^{\mathrm{vect}}_{[0,1]}(X)$ that consists of admissible prismatic F-crystals of height $[0,1]$. We then establish an equivalence of the above category with yet another category.

As a preparation, we show that the filtration on an F-gauge in vector bundles is always saturated.

Proposition 3.52. *Let S be a quasiregular semiperfectoid ring, and let $E = (E, \mathrm{Fil}^\bullet E, \widetilde{\varphi}_E)$ be an F-gauge in vector bundles over S. Then E is saturated, or more explicitly we have $\mathrm{Fil}^iE = \widetilde{\varphi}_E^{-1}(I^iE)$ for $i \in \mathbb{Z}$.*

Corollary 3.53. *Let X be a quasi-syntomic p-adic formal scheme. Then the forgetful functor below from F-gauges in vector bundles to its underlying F-crystals is fully faithful*

$$\mathrm{F\text{-}Gauge}^{\mathrm{vect}}(X) \longrightarrow \mathrm{F\text{-}Crys}^{\mathrm{vect}}(X).$$

Proof. By assumption, as $\mathrm{Fil}^\bullet E$ is a filtered vector bundle over the filtered ordinary ring $\mathrm{Fil}^\bullet_N \Delta_S$, each Fil^iE is a submodule inside E such that $\widetilde{\varphi}_E(\mathrm{Fil}^iE) \subset I^iE$. Thus $\mathrm{Fil}^iE \subseteq \widetilde{\varphi}_E^{-1}(I^iE)$, and we want to show that each inclusion is an equality. Here we note that by the assumption of $\mathrm{Fil}^\bullet E$, we know the inclusion is an equality when $i << 0$.

For each $i \in \mathbb{Z}$, the Frobenius structure $\widetilde{\varphi}_E$ induces the following commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Fil}^{i+1}E \arrow[r] \arrow[d, "\widetilde{\varphi}_E^{i+1}"] & \mathrm{Fil}^iE \arrow[r] \arrow[d, "\widetilde{\varphi}_E^{i}"] & \mathrm{Gr}^iE \arrow[r] \arrow[d, "\mathrm{Gr}^i\widetilde{\varphi}"] & 0 \\
0 \arrow[r] & I^{i+1}E \arrow[r] & I^iE \arrow[r] & I^iE/I^{i+1}E \arrow[r] & 0,
\end{tikzcd}$$

where $\widetilde{\varphi}_E^i$ is the restriction of $\widetilde{\varphi}_E$ on Fil^iE. We then claim that the map $\mathrm{Gr}^iE \to I^iE/I^{i+1}E$ is injective for each $i \in \mathbb{Z}$. Granting the claim, a simple diagram chasing implies that $\mathrm{Fil}^{i+1}E = (\widetilde{\varphi}_E^i)^{-1}(I^{i+1}E) = \widetilde{\varphi}_E^{-1}(I^{i+1}E)$. Hence we get the saturatedness of $\mathrm{Fil}^\bullet E$ by induction and the observation that $\mathrm{Fil}^iE = E$ for $i << 0$.

Now we check the above claim. Recall that the graded map $\mathrm{Gr}^\bullet\widetilde{\varphi} : \mathrm{Gr}^\bullet E \to \bar{I}^\bullet E$ factors through the linearlization

$$\mathrm{Gr}^\bullet E \longrightarrow \mathrm{Gr}^\bullet E \widehat{\bigotimes}_{\mathrm{Gr}^\bullet_N \Delta_S, \mathrm{Gr}^\bullet \varphi_{\Delta_S}} \bar{I}^{\mathbb{N}}\overline{\Delta}_S \xrightarrow{\mathrm{Gr}^\bullet \varphi_E} \bar{I}^\bullet E,$$

such that a further base change to $\bar{I}^{\mathbb{Z}}\overline{\Delta}_S$ is a graded isomorphism (cf. Definition 3.20). In particular, to show the injection of $\mathrm{Gr}^iE \to \bar{I}^iE$, it suffices to show that the following map is a graded injection

$$\mathrm{Gr}^\bullet E \longrightarrow \mathrm{Gr}^\bullet E \widehat{\bigotimes}_{\mathrm{Gr}^\bullet_N \Delta_S, \mathrm{Gr}^\bullet \varphi_{\Delta_S}} \bar{I}^{\mathbb{Z}}\overline{\Delta}_S.$$

We then notice that since $\mathrm{Gr}^\bullet E$ is a graded vector bundle over $\mathrm{Gr}^\bullet_N \Delta_S$, its reduction $\mathrm{Red}_S(E)$, which is the graded base change of $\mathrm{Gr}^\bullet E$ from $\mathrm{Gr}^\bullet_N \Delta_S$ to S, is a graded vector bundle over S. Moreover, by Theorem 3.45, the complex $\mathrm{Gr}^\bullet E$ is filtered by a finite exhaustive weight filtration with graded pieces being

[7]Although the authors only stated that admissibility is automatic for complete regular local rings with a perfect characteristic p residue field, their proof works for any p-complete ring R admitting a quasi-syntomic cover by a perfectoid.

$\mathrm{Red}_j(E) \otimes_S \mathrm{Gr}^\bullet_N \Delta_S$. Thus by a finite induction, it suffices to assume $\mathrm{Gr}^\bullet E$ is of the form $V \otimes_S \mathrm{Gr}^\bullet_N \Delta_S$ for a finite projective S-module V.

Finally, as S is quasiregular semiperfectoid, by [BS22, Thm. 12.2] the graded map $\mathrm{Gr}^\bullet \varphi_{\Delta_S} : \mathrm{Gr}^\bullet_N \Delta_S \to \overline{I}^{\mathbb{N}} \overline{\Delta}_S$ and thus $\mathrm{Gr}^\bullet_N \Delta_S \to \overline{I}^{\mathbb{Z}} \overline{\Delta}_S$ is injective. So by the flatness of V over S, we get the injection of

$$V \otimes_S \mathrm{Gr}^i_N \Delta_S \longrightarrow \mathrm{Gr}^i \left((V \otimes_S \mathrm{Gr}^\bullet_N \Delta_S) \widehat{\bigotimes_{\mathrm{Gr}^\bullet_N \Delta_S, \mathrm{Gr}^\bullet \varphi_{\Delta_S}}} \overline{I}^{\mathbb{Z}} \overline{\Delta}_S \right) \simeq V \otimes_S \overline{I}^i \overline{\Delta}_S,$$

hence the claim. $\square$

Theorem 3.54. *Let X be a quasi-syntomic p-adic formal scheme. There is a natural equivalence*

$$\text{F-Gauge}^{\mathrm{vect}}_{[0,1]}(X) \simeq \text{F-Crys}^{\mathrm{adm},\mathrm{vect}}_{[0,1]}(X).$$

It is compatible with Theorem 3.32 when X is smooth over $\mathrm{Spf}(\mathcal{O}_K)$.

The subscript $[0,1]$ denotes objects which are effective of height ≤ 1.

Proof. Let $(\mathcal{E}, \varphi_{\mathcal{E}})$ be a given object in $\text{F-Crys}^{\mathrm{adm},\mathrm{vect}}_{[0,1]}(X)$. We use $(M_S, \mathrm{Fil}^\bullet M_S, \widetilde{\varphi}_{M_S})$ to denote the value of $\Pi_{\mathrm{Spf}(S)}(\mathcal{E}|_{\mathrm{Spf}(S)_{\mathrm{qrsp}}})$ at $S \in X_{\mathrm{qrsp}}$, which by Theorem 3.32 (1) satisfies

$$\mathrm{Fil}^n M_S = \widetilde{\varphi}^{-1}_{M_S}(I^n M_S) \subset M_S.$$

We then claim that

1. the filtered module $\mathrm{Fil}^\bullet M_S$ is a filtered vector bundle over $\mathrm{Fil}^\bullet_N \Delta_S$, and
2. for a map of rings $S_1 \to S_2$ in X_{qrsp}, the natural filtered linearlization map below is a filtered isomorphism
$$\mathrm{Fil}^\bullet M_1 \otimes_{\mathrm{Fil}^\bullet_N \Delta_{S_1}} \mathrm{Fil}^\bullet_N \Delta_{S_2} \longrightarrow \mathrm{Fil}^\bullet M_2.$$

Here we use M_i to simplify the notation M_{S_i}. Granting the above two claims, we see every admissible prismatic F-crystal in vector bundles of height $[0,1]$ naturally underlies an F-gauge in vector bundles of weight $[0,1]$. Conversely, given an F-gauge in vector bundles of weight $[0,1]$, its underlying F-crystal is an F-crystal in vector bundle of height $[0,1]$ by Proposition 3.51.(i).

Since both statements can be checked locally on S, we are allowed to localize S. By doing so, we may first assume that the ideal I in Δ_S is generated by d. By [ALB23, Prop. 4.29, Lem. 4.23, Prop. 4.22] (applied at $(A, \mathrm{Fil} A, \varphi, \varphi_1) = (\Delta_S, \mathrm{Fil}^1_N \Delta_S, \varphi_{\Delta_S}, \frac{1}{d}\varphi_{\Delta_S})$, which satisfies the assumption of [ALB23, Prop. 4.22] by loc. cit. Lem 4.27, Lem. 4.28), there are finite projective Δ_S-modules L_0 and L_1, such that

$$M_S \simeq L_0 \oplus L_1, \quad \mathrm{Fil}^1 M_S \simeq \mathrm{Fil}^1_N \Delta_S \otimes_{\Delta_S} L_0 \oplus L_1.$$

Moreover, as in loc. cit., if we localize on S so that L_0 and L_1 are finite free and choose a basis e_i (resp. f_j) of L_0 (resp. L_1), then we know that $\{\varphi(e_i), \varphi_1(f_j)\}$ is a basis of M. This implies that as a filtered module over $\mathrm{Fil}^\bullet_N \Delta_S$, we have

$$\mathrm{Fil}^\bullet M = \mathrm{Fil}^\bullet_N \Delta_S \otimes_{\Delta_S} L_0 \oplus \mathrm{Fil}^{\bullet-1}_N \Delta_S \otimes_{\Delta_S} L_1.$$

For (2), we apply (1) at $S = S_1$, to get the aforementioned decomposition. By the same reasoning as before, we see $\mathrm{Fil}^\bullet M_2 \simeq \mathrm{Fil}^\bullet_N \Delta_{S_2} \otimes_{\Delta_{S_1}} L_0 \oplus \mathrm{Fil}^{\bullet-1}_N \Delta_{S_2} \otimes_{\Delta_{S_1}} L_1$, and the natural filtered linearization $\mathrm{Fil}^\bullet_N \Delta_{S_2} \otimes_{\mathrm{Fil}^\bullet_N \Delta_{S_1}} \mathrm{Fil}^\bullet M_1 \to \mathrm{Fil}^\bullet M_2$ is a filtered isomorphism.

Finally we show the essential surjectivity. By Proposition 3.52, an F-gauge in vector bundle has the saturated Nygaardian filtration and is uniquely determined by the underlying F-crystal. So it is left to check that the underlying F-crystal is admissible in the sense of [ALB23, Def. 4.5, Def. 4.10]. Namely for $S \in X_{\mathrm{qrsp}}$, we want to show that the image of $M_S \xrightarrow{\widetilde{\varphi}_{M_S}} M_S \to M_S/IM_S$ is a vector bundle F_S over S, such that the linearization $F_S \otimes_S \overline{\Delta}_S \to M_S/IM_S$ is injective. By Proposition 3.52, $\mathrm{Fil}^1 M_S$ is equal to $\widetilde{\varphi}^{-1}(IM_S)$, and thus the image F_S is equal to $M_S/\mathrm{Fil}^1 M_S = \mathrm{Gr}^0 M_S$. The zero-th graded piece $\mathrm{Gr}^0 M_S = F_S$ is a vector bundle over $\mathrm{Gr}^0_N \Delta_S = S$, thanks to the assumption that $\mathrm{Fil}^\bullet M_S$ is a filtered vector bundle over $\mathrm{Fil}^\bullet_N \Delta_S$. Moreover, by taking the zero-th graded piece of the filtered isomorphism $\varphi_{M_S} : \mathrm{Fil}^\bullet M_S \otimes_{\mathrm{Fil}^\bullet_N \Delta_S, \varphi_{\Delta_S}} I^{\mathbb{Z}}\Delta_S \simeq I^{\mathbb{Z}} M_S$, the required injectivity of the linearlization above is equivalent to the injectivity of the map

$$\mathrm{Gr}^0 M_S \otimes_S \overline{\Delta}_S \longrightarrow \mathrm{Gr}^0\left(\mathrm{Fil}^\bullet M_S \otimes_{\mathrm{Fil}^\bullet_N \Delta_S, \varphi_{\Delta_S}} I^{\mathbb{Z}}\Delta_S\right).$$

Here by an induction on the weight filtration, we may assume $\mathrm{Gr}^\bullet M_S = V \otimes_S \mathrm{Gr}^\bullet_N \Delta_S$ for a finite projective S-module V of graded degree 0 or 1. In both cases, the injection follows from that of the structure sheaf, which finishes the proof. □

4. Relative prismatic F-gauges and realizations

We now turn our focus to a relative version of prismatic F-gauges, and its relation with the absolute version studied in the previous section. Throughout this section, let us fix (A, I) a bounded prism with $\overline{A} := A/I$.

4.1. **Quasi-syntomic sheaves in relative setting.** Let Z be a quasi-syntomic $\overline{A}$-formal scheme. Consider $(Z/\overline{A})_{\mathrm{qrsp}} \subset Z_{\mathrm{qSyn}}$, where $(Z/\overline{A})$qrsp denote the category of large quasi-syntomic $\overline{A}$-algebras (in the sense of [BS22, Definition 15.1]) together with a quasi-syntomic map to Z equipped with quasi-syntomic topology. Then we have the following analog of the "*unfolding process*" in [BMS19, Proposition 4.31]:

Proposition 4.1. *Restriction to qrsp objects induces an equivalence*

$$\mathrm{Shv}_{\mathcal{C}}(Z_{\mathrm{qSyn}}) \cong \mathrm{Shv}_{\mathcal{C}}((Z/\overline{A})_{\mathrm{qrsp}})$$

for any presentable ∞-category $\mathcal{C}$.

Proof. The proof is identical to that in loc. cit. □

Remark 4.2. Concretely, suppose we are given a quasi-syntomic sheaf $\mathcal{F}$ on $(Z/\overline{A})_{\mathrm{qrsp}}$. Let R be an object in Z_{qSyn}, and suppose we are given a large quasi-syntomic cover $R \to S^{(0)}$ with its Cech nerve being $S^{(\bullet)}$, then the unfolding evaluated at R can be computed as the cosimplicial limit of $\mathcal{F}(S^{(\bullet)})$.

Example 4.3 (see [BMS19, §1.3 and §5.2] and [BS22, §15])**.** Here are some examples of quasi-syntomic sheaves on $\mathrm{Spf}(\overline{A})_{\mathrm{qSyn}}$: the relative prismatic cohomology $\Delta_{-/A}$ and its φ_A^*-variant $\Delta^{(1)}_{-/A} := \Delta_{-/A}\widehat{\otimes}_{A,\varphi_A} A$, the i-th Nygaard filtration on $\Delta^{(1)}_{-/A}$ for any i, the relative Hodge–Tate cohomology $\overline{\Delta}_{-/A} := \Delta_{-/A}/I$ and its i-th conjugate filtration for any i, the p-adic derived de Rham cohomology $\mathrm{dR}_{-/\overline{A}}$ and its i-th Hodge filtration for any i. By restriction, we get quasi-syntomic sheaves on Z_{qSyn}.

The following relation between the various filtrations appearing above will often be used (see [BS22, Theorem 15.2 and 15.3]): we have fiber sequences

$$\mathrm{Fil}^{i+1}_N \Delta^{(1)}_{-/A} \to \mathrm{Fil}^i_N \Delta^{(1)}_{-/A} \to \mathrm{Gr}^i_N \Delta^{(1)}_{-/A} \cong \mathrm{Fil}^{\mathrm{conj}}_i \overline{\Delta}_{-/A}\{i\};\ \text{and}$$

$$\mathrm{Fil}^{i-1}_N \Delta^{(1)}_{-/A} \otimes_A I \to \mathrm{Fil}^i_N \Delta^{(1)}_{-/A} \to \mathrm{Fil}^i_H \mathrm{dR}_{-/\overline{A}}.$$

In terms of Rees's construction, we can formulate them uniformly:

Proposition 4.4. *For simplicity, let us assume $(A, I = (d))$ is a bounded oriented prism. Now we view $\mathrm{Rees}(\mathrm{Fil}^\bullet_N \Delta^{(1)}_{-/A}) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i_N \Delta^{(1)}_{-/A} \cdot t^{-i}$ as a graded $\mathrm{Rees}(I^{\mathbb{N}}A) := A[t, \frac{d}{t}]$-algebra. Then we have*

$$\mathrm{Rees}(\mathrm{Fil}^\bullet_N \Delta^{(1)}_{-/A})/^L(t) \cong \bigoplus_{i\in\mathbb{N}} \mathrm{Fil}^{\mathrm{conj}}_i \overline{\Delta}_{-/A} \cdot (\frac{d}{t})^i;\ \textit{and}$$

$$\mathrm{Rees}(\mathrm{Fil}^\bullet_N \Delta^{(1)}_{-/A})/^L(\frac{d}{t}) \cong \mathrm{Rees}(\mathrm{Fil}^\bullet_H \mathrm{dR}_{-/\overline{A}}) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i_H \mathrm{dR}_{-/\overline{A}} \cdot t^{-i}.$$

The following flatness result is the analog of (proof of) Proposition 3.30.

Proposition 4.5. *Let $S_1 \to S_2$ be a quasi-syntomic flat (resp. faithfully flat) map of two quasi-syntomic $\overline{A}$-algebras and let $S_1 \to S_3$ be an arbitrary map of quasi-syntomic $\overline{A}$-algebras with $S_4 := S_2\widehat{\otimes}_{S_1} S_3$, then:*

(1) The map $\mathrm{Hdg}(S_1/\overline{A}) \to \mathrm{Hdg}(S_2/\overline{A})$ *is p-completely flat (resp. faithfully flat), and the graded map*

$$\mathrm{Hdg}(S_2/\overline{A})\widehat{\otimes}_{\mathrm{Hdg}(S_1/\overline{A})}\mathrm{Hdg}(S_3/\overline{A}) \to \mathrm{Hdg}(S_4/\overline{A})$$

is a graded isomorphism;

(2) The filtered map $\mathrm{Fil}^{\mathrm{conj}}_{\bullet}(\overline{\Delta}_{S_1/A}) \to \mathrm{Fil}^{\mathrm{conj}}_{\bullet}(\overline{\Delta}_{S_2/A})$ *has its corresponding Rees's construction map being p-completely flat (resp. faithfully flat), and the filtered map*

$$\mathrm{Fil}^{\mathrm{conj}}_{\bullet}(\overline{\Delta}_{S_2/\overline{A}})\widehat{\otimes}_{\mathrm{Fil}^{\mathrm{conj}}_{\bullet}(\overline{\Delta}_{S_1/\overline{A}})}\mathrm{Fil}^{\mathrm{conj}}_{\bullet}(\overline{\Delta}_{S_3/\overline{A}}) \to \mathrm{Fil}^{\mathrm{conj}}_{\bullet}(\overline{\Delta}_{S_4/\overline{A}})$$

is a filtered isomorphism;

(3) The filtered map $\mathrm{Fil}^{\bullet}_{H}(\mathrm{dR}_{S_1/\overline{A}}) \to \mathrm{Fil}^{\bullet}_{H}(\mathrm{dR}_{S_2/\overline{A}})$ *has its corresponding Rees's construction map being p-completely flat (resp. faithfully flat), and the filtered map*

$$\mathrm{Fil}^{\bullet}_{H}(\mathrm{dR}_{S_2/\overline{A}})\widehat{\otimes}_{\mathrm{Fil}^{\bullet}_{H}(\mathrm{dR}_{S_1/\overline{A}})}\mathrm{Fil}^{\bullet}_{H}(\mathrm{dR}_{S_3/\overline{A}}) \to \mathrm{Fil}^{\bullet}_{H}(\mathrm{dR}_{S_4/\overline{A}})$$

is a filtered isomorphism;

(4) The filtered map $\mathrm{Fil}^{\bullet}_{N}(\Delta^{(1)}_{S_1/A}) \to \mathrm{Fil}^{\bullet}_{N}(\Delta^{(1)}_{S_2/A})$ *has its corresponding Rees's construction map being* (p, I)*-completely flat (resp. faithfully flat), and the filtered map*

$$\mathrm{Fil}^{\bullet}_{N}(\Delta^{(1)}_{S_2/A})\widehat{\otimes}_{\mathrm{Fil}^{\bullet}_{N}(\Delta^{(1)}_{S_1/A})}\mathrm{Fil}^{\bullet}_{N}(\Delta^{(1)}_{S_3/A}) \to \mathrm{Fil}^{\bullet}_{N}(\Delta^{(1)}_{S_4/A})$$

is a filtered isomorphism.

Proof. The proof follows from similar argument appearing in the proof of Proposition 3.30: the proof of (1) has appeared there. Then similar to there, with Proposition 4.4 in mind, we can use (1) together with Lemma 3.28 and Lemma 3.29 to prove (2), and use (2) and Lemma 3.28 to prove both of (3) and (4). ☐

4.2. **Relative prismatic F-gauges.** Let (A, I) be a fixed bounded prism. For a quasi-syntomic $\overline{A}$-formal scheme Z, we denote by $(Z/\overline{A})_{\mathrm{qrsp}}$ the category of large quasi-syntomic $\overline{A}$-algebra over Z together with quasi-syntomic topology, in the sense of [BS22, Definition 15.1].

Definition 4.6 (c.f. [Tan22, Definition 2.14])**.** Let S be a large quasi-syntomic $\overline{A}$-algebra. A *prismatic gauge* $E = (E, \mathrm{Fil}^{\bullet}E^{(1)})$ over $\mathrm{Spf}(S)/A$ consists of the following data:

- a derived (p, I)-complete complex E over $\Delta_{S/A}$;
- a derived (p, I)-complete filtration $\mathrm{Fil}^{\bullet}E^{(1)}$ on $E^{(1)} := \varphi_A^* E$, linear over the Nygaard filtered relative prismatic cohomology $\mathrm{Fil}^{\bullet}_{N}\Delta^{(1)}_{S/A}$.

A *prismatic F-gauge* $E = (E, \mathrm{Fil}^{\bullet}E^{(1)}, \widetilde{\varphi}_E)$ over $\mathrm{Spf}(S)/A$ consists of the following data:

- a prismatic gauge $E = (E, \mathrm{Fil}^{\bullet}E^{(1)})$ over $\Delta_{S/A}$ called the underlying gauge of E;
- a map of filtered complexes
$$\widetilde{\varphi}_E \colon \mathrm{Fil}^{\bullet}E^{(1)} \longrightarrow I^{\mathbb{Z}}E = I^{\mathbb{Z}}\Delta_{S/A}\widehat{\otimes}_{\Delta_{S/A}}E,$$
such that the filtered linearization
$$\varphi_E \colon \mathrm{Fil}^{\bullet}E^{(1)}\widehat{\otimes}_{\mathrm{Fil}^{\bullet}_{N}\Delta^{(1)}_{S/A}, \varphi_{\Delta_S}} I^{\mathbb{Z}}\Delta_S \longrightarrow I^{\mathbb{Z}}E$$
is a filtered isomorphism in $\mathcal{DF}_{(p,I)\text{-comp}}(I^{\mathbb{Z}}\Delta_{S/A}) \simeq \mathcal{D}_{(p,I)\text{-comp}}(\Delta_{S/A})$.

We use (F-)Gauge($\mathrm{Spf}(S)/A$) to denote the natural ∞-category whose objects are as above.

Remark 4.7 (c.f. [Tan22, Proposition 2.1 and Remark 2.5])**.** Here we notice that by [BS22, Thm. 15.2], the filtered morphism of A-algebras

$$\varphi_{\Delta_S} \colon \mathrm{Fil}^{\bullet}_{N}\Delta^{(1)}_{S/A} \to I^{\bullet}\Delta_{S/A}$$

identifies the target as (p, I)-completely inverting $I \subset \mathrm{Fil}^1_N$ of the source as filtered A-algebras. In particular, the above linarization condition can be rewritten as a filtered isomorphism

$$\left(\mathrm{Fil}^{\bullet}E^{(1)}[1/I]\right)^{\wedge}_{(p,I)} \xrightarrow[\cong]{\varphi_E} I^{\mathbb{Z}}E,$$

or equivalently

$$\Big(\mathrm{colim}_{i\in\mathbb{Z}}(\cdots \to \mathrm{Fil}^i E^{(1)} \otimes I^{-i} \to \mathrm{Fil}^{i+1} E^{(1)} \otimes I^{-i-1} \to \cdots)\Big)^{\wedge}_{(p,I)} \xrightarrow[\cong]{\varphi_E} E.$$

Definition 4.8. As in Definition 3.21, we say a prismatic F-gauge $E = (E, \mathrm{Fil}^\bullet E^{(1)}, \widetilde{\varphi}_E)$ over $\mathrm{Spf}(S)/A$ is:

(1) *finite projective* if E is (p,I)-completely finite projective over $\Delta_{S/A}$ and $\mathrm{Fil}^\bullet E^{(1)}$ is filtered (p,I)-completely finite projective over $\mathrm{Fil}^\bullet_N \Delta^{(1)}_{S/A}$;

(2) *perfect* if E is (p,I)-completely perfect over $\Delta_{S/A}$ and $\mathrm{Fil}^\bullet E^{(1)}$ is filtered (p,I)-completely perfect over $\mathrm{Fil}^\bullet_N \Delta^{(1)}_{S/A}$;

(3) *coherent* if it is perfect with all of E, $\mathrm{Fil}^\bullet E^{(1)}$ and $\mathrm{Gr}^\bullet E^{(1)}$ concentrated in cohomological degree 0.

We use (F-)Gauge$^*(\mathrm{Spf}(S)/A)$ to denote the natural ∞-category whose objects are as above, where $* \in \{\mathrm{vect}, \mathrm{perf}, \mathrm{coh}\}$ respectively.

The analog of Proposition 3.23 still holds true:

Proposition 4.9. *Let S be a large quasi-syntomic $\overline{A}$-algebra and let $(E, \mathrm{Fil}^\bullet E^{(1)}) \in \mathrm{Gauge}(\mathrm{Spf}(S)/A)$. Then, when making the Definition 4.8.(1)-(2), we may drop (p,I)-completely from the condition without changing the definition.*

Proof. The part concerning E follows from Proposition 2.22. The part concerning $\mathrm{Fil}^\bullet E^{(1)}$ follows from the same proof of Proposition 3.23, as long as we can show the following variant of [ALB23, Lemma 4.28]: The pair $(\Delta^{(1)}_{S/A}, \mathrm{Fil}^1_N \Delta^{(1)}_{S/A})$ is henselian. As $\Delta^{(1)}_{S/A}$ is I-adically complete, we may check the claim after modulo I. By the discussion before Proposition 4.4, we are reduced to showing that $(\mathrm{dR}_{S/\overline{A}}, \mathrm{Fil}^1_H \mathrm{dR}_{S/\overline{A}})$ is a henselian pair. Our claim now follows from the fact that $\mathrm{dR}_{S/\overline{A}}$ is p-adically complete and the fact that Fil^1_H always admits a (derived) divided power structure, see [Mag24, p. 5 and §4]. $\square$

Remark 4.10. Similar to Remark 3.24, for a map of large quasi-syntomic $\overline{A}$-algebras $S_1 \to S_2$ and $* \in \{\emptyset, \mathrm{vect}, \mathrm{perf}\}$, the completed filtered base change induces a natural functor

$$\Phi_{(S_1,S_2)} : \text{(F-)Gauge}^*(\mathrm{Spf}(S_1)/A) \longrightarrow \text{(F-)Gauge}^*(\mathrm{Spf}(S_2)/A),$$

$$(E,\ \mathrm{Fil}^\bullet E^{(1)}, (\varphi_E)) \longmapsto (E \widehat{\bigotimes}_{\Delta_{S_1/A}} \Delta_{S_2/A}, \mathrm{Fil}^\bullet E^{(1)} \widehat{\bigotimes}_{\mathrm{Fil}^\bullet_N \Delta^{(1)}_{S_1/A}} \mathrm{Fil}^\bullet_N \Delta^{(1)}_{S_2/A}, (\widetilde{\varphi}_E \otimes_{\varphi_{\Delta_{S_1/A}}} \varphi_{\Delta_{S_2/A}})).$$

Definition 4.11. Let Z be a quasi-syntomic $\overline{A}$-formal scheme. The category of *prismatic (F-)gauges* over Z/A is defined as the limit of ∞-categories

$$\text{(F-)Gauge}^*(Z/A) := \lim_{S \in (Z/\overline{A})_{\mathrm{qrsp}}} \text{(F-)Gauge}^*(\mathrm{Spf}(S)/A),$$

where $* \in \{\emptyset, \mathrm{vect}, \mathrm{perf}, \mathrm{coh}\}$.

As in the absolute case (Proposition 3.30), we have the following sheaf property.

Proposition 4.12. *Let $S \to S^{(0)}$ be a quasi-syntomic cover of large quasi-syntomic $\overline{A}$-algebras, and let $S^{(\bullet)}$ be the p-completed Čech nerve. Then for $* \in \{\emptyset, \mathrm{perf}, \mathrm{vect}\}$, we have a natural equivalence*

$$\text{(F-)Gauge}^*(\mathrm{Spf}(S)/A) \simeq \lim_{[n]\in\Delta} \text{(F-)Gauge}^*(\mathrm{Spf}(S^{(n)})/A).$$

Proof. The proof strategy is exactly the same as that of Proposition 3.30: via Rees's dictionary, the descent property follows from Proposition 4.5 (4). $\square$

Our main construction in this section is the following.

Theorem 4.13. *Let X be a quasi-syntomic formal scheme, and let Z be a quasi-syntomic $\overline{A}$-formal scheme, together with a map $Z \to X$. Then for $* \in \{\emptyset, \mathrm{perf}, \mathrm{vect}\}$, there is a natural functor*

$$BC_{X,Z/A} \colon \text{(F-)Gauge}^*(X) \longrightarrow \text{(F-)Gauge}^*(Z/A),$$

compatible with the natural base change functor of prismatic F-crystals (F-)Crys$^*(X) \to$ (F-)Crys$^*(Z/A)$.

As a preparation, we first consider the special case of quasiregular semiperfectoid rings.

Lemma 4.14. *Let S be a quasiregular semiperfectoid ring, and let S' a large quasi-syntomic $\overline{A}$-algebra, together with a map $\mathrm{Spf}(S') \to \mathrm{Spf}(S)$. Then there is a functorial commutative diagram of filtered rings*

$$\begin{tikzcd} \mathrm{Fil}^\bullet_N {\mathbb{\Delta}}_S \arrow[r] \arrow[d, "\varphi_{{\mathbb{\Delta}}_S}"] & \mathrm{Fil}^\bullet_N {\mathbb{\Delta}}^{(1)}_{S'/A} \arrow[d, "\varphi_{{\mathbb{\Delta}}^{(1)}_{S'/A}}"] \\ I^\bullet {\mathbb{\Delta}}_S \arrow[r] & I^\bullet {\mathbb{\Delta}}_{S'/A}, \end{tikzcd}$$

such that the top row underlies the composition of homomorphisms of rings ${\mathbb{\Delta}}_S \to {\mathbb{\Delta}}_{S'/A} \to {\mathbb{\Delta}}^{(1)}_{S'/A}$.

Proof. There is a natural commutative diagram:

$$\begin{tikzcd} {\mathbb{\Delta}}_S \arrow[r] \arrow[dd, "\varphi_{{\mathbb{\Delta}}_S}"] & {\mathbb{\Delta}}_{S'/A} \arrow[d, "\mathrm{id}\otimes 1"] \\ & {\mathbb{\Delta}}^{(1)}_{S'/A} := {\mathbb{\Delta}}_{S'/A} \widehat{\otimes}_{A,\varphi_A} A \arrow[d, "\varphi_{rel}"] \\ {\mathbb{\Delta}}_S \arrow[r] & {\mathbb{\Delta}}_{S'/A}, \end{tikzcd}$$

where the horizontal maps follow from the initial property of ${\mathbb{\Delta}}_S$ in the category $\mathrm{Spf}(S)_{\mathbb{\Delta}}$, and the right vertical map is the natural factorization of $\varphi_{{\mathbb{\Delta}}_{S'/A}}$, with φ_{rel} being the relative Frobenius map for prismatic cohomology of S' over A. Moreover, if an element $x \in {\mathbb{\Delta}}_S$ is sent into $I^n {\mathbb{\Delta}}_S$ under $\varphi_{{\mathbb{\Delta}}_S}$, then by the commutativity, the image of x in ${\mathbb{\Delta}}^{(1)}_{S'/A}$ is sent into $I^n {\mathbb{\Delta}}_{S'/A}$ under φ_{rel}. By definition of Nygaard filtration for quasiregular semiperfectoid rings [BS22, Theorem 15.2.(1)], this in particular means that the above diagram carries $\mathrm{Fil}^n_N {\mathbb{\Delta}}_S$ into $\mathrm{Fil}^n_N {\mathbb{\Delta}}^{(1)}_{S'/A}$ under the top right composition:

$${\mathbb{\Delta}}_S \longrightarrow {\mathbb{\Delta}}_{S'/A} \xrightarrow{\mathrm{id}\otimes 1} {\mathbb{\Delta}}^{(1)}_{S'/A}.$$

Thus we get the filtered map as needed. □

Proof of Theorem 4.13. We prove the version with F, the one without F is similar and easier. Assume $S \in X_{\mathrm{qrsp}}$ is any quasiregular semiperfectoid ring over X. Then since $\mathrm{Spf}(S) \times_X Z$ is a quasi-syntomic formal scheme over Z, and Z is quasi-syntomic over $\mathrm{Spf}(\overline{A})$, we see $\mathrm{Spf}(S) \times_X Z$ admits quasi-syntomic covers by formal spectra of large quasi-syntomic $\overline{A}$-algebras. Let S' be any large quasi-syntomic $\overline{A}$-algebra that is quasi-syntomic over $\mathrm{Spf}(S) \times_X Z$. Since both F-gauges and relative F-gauges satisfy quasi-syntomic descent (Proposition 3.30 and Proposition 4.12), it suffices to construct functors $BC_{(S,S')} \colon \text{F-Gauge}^*(\mathrm{Spf}(S)) \longrightarrow \text{F-Gauge}^*(\mathrm{Spf}(S')/A)$ compatible with base change in both S and S' (defined in Remark 3.24 and Remark 4.10 respectively).

Lastly we define the functor $BC_{(S,S')}$, for any pair (S, S') as above, by taking the filtered base change:

$$\begin{aligned} BC_{(S,S')} \colon &\text{F-Gauge}^*(\mathrm{Spf}(S)) \longrightarrow \text{F-Gauge}^*(\mathrm{Spf}(S')/A), \\ &E = (E, \mathrm{Fil}^\bullet E, \varphi_E) \longmapsto \left(E \widehat{\bigotimes}_{{\mathbb{\Delta}}_S} {\mathbb{\Delta}}_{S'/A}, \mathrm{Fil}^\bullet E \widehat{\bigotimes}_{\mathrm{Fil}^\bullet_N {\mathbb{\Delta}}_S} \mathrm{Fil}^\bullet_N {\mathbb{\Delta}}^{(1)}_{S'/A}, \widetilde{\varphi}_E \otimes \varphi_{{\mathbb{\Delta}}^{(1)}_{S'/A}} \right). \end{aligned}$$

Here, unraveling definitions, the filtered map $\widetilde{\varphi}_E \otimes \varphi_{{\mathbb{\Delta}}^{(1)}_{S'/A}}$ is given by

$$\mathrm{Fil}^\bullet E \widehat{\bigotimes}_{\mathrm{Fil}^\bullet_N {\mathbb{\Delta}}_S} \mathrm{Fil}^\bullet_N {\mathbb{\Delta}}^{(1)}_{S'/A} \xrightarrow{\widetilde{\varphi}_E \otimes_{\varphi_{{\mathbb{\Delta}}_S}} \varphi_{{\mathbb{\Delta}}^{(1)}_{S'/A}}} I^\bullet E \widehat{\bigotimes}_{I^\bullet {\mathbb{\Delta}}_S} I^\bullet {\mathbb{\Delta}}_{S'/A} \simeq I^\bullet (E \widehat{\otimes}_{{\mathbb{\Delta}}_S} {\mathbb{\Delta}}_{S'/A}),$$

and by the commutative diagram in Lemma 4.14, its linearization is

$$\left(\mathrm{Fil}^\bullet E\widehat{\bigotimes}_{\mathrm{Fil}^\bullet_N \Delta_S}\mathrm{Fil}^\bullet_N \Delta^{(1)}_{S'/A}\right)\widehat{\bigotimes}_{\mathrm{Fil}^\bullet_N \Delta^{(1)}_{S'/A}} I^\bullet \Delta_{S'/A} \cong \left(\mathrm{Fil}^\bullet E\widehat{\bigotimes}_{\mathrm{Fil}^\bullet_N \Delta_S} I^\bullet \Delta_S\right)\widehat{\bigotimes}_{I^\bullet \Delta_S} I^\bullet \Delta_{S'/A}$$
$$\xrightarrow{\varphi_E\otimes \mathrm{id}} I^\bullet E\widehat{\bigotimes}_{I^\bullet \Delta_S} I^\bullet \Delta_{S'/A},$$

which is a filtered isomorphism by assumption of $E=(E,\mathrm{Fil}^\bullet E,\varphi_E)\in$ F-Gauge(Spf(S)). $\square$

Remark 4.15. Let us note that the functor in Theorem 4.13 satisfies base change with respect to the base prism: in the setting of the theorem, suppose we are given a map of prisms $(A,I)\to(\widetilde{A},I\widetilde{A})$ and let $\widetilde{Z}:=Z\times_{\overline{A}}\overline{\widetilde{A}}$. Then we have a commutative diagram:

$$\begin{tikzcd}
\text{(F-)Gauge}^*(X) \arrow[rr] \arrow[dr] & & \text{(F-)Gauge}^*(Z/A) \arrow[dl, "-\widehat{\otimes}_A\widetilde{A}"] \\
& \text{(F-)Gauge}^*(\widetilde{Z}/\widetilde{A}), &
\end{tikzcd}$$

where the two unlabelled arrows are what constructed in Theorem 4.13. Indeed, tracing through the construction in the proof of Theorem 4.13, we are reduced to a similar statement with X and Z replaced by S and S' (and $\widetilde{Z}$ replaced by $\widetilde{S'}:=S'\widehat{\otimes}_{\overline{A}}\overline{\widetilde{A}}$), which immediately follows from the base change formula of prismatic cohomology

$$\Delta_{S'/A}\widehat{\otimes}_A\widetilde{A}\cong\Delta_{\widetilde{S'}/\widetilde{A}},$$

as well as similar base change formulas for Nygaard filtrations ([BS22, Theorem 15.2.(3)]).

Similar to Section 3.3, there is a natural notion of weight and weight filtration on the associated graded of a relative prismatic F-gauge.

Definition 4.16. Let Z be a quasi-syntomic $\overline{A}$-formal scheme, and let $[a,b]$ be an interval in $\mathbb{R}\cup\{-\infty,\infty\}$. For a graded complex $M^\bullet\in\mathcal{DG}^*_{p\text{-comp}}((Z/\overline{A})_{\mathrm{qrsp}},\mathrm{Gr}^\bullet_N\Delta^{(1)}_{-/A})$.

(1) The reduction of $M^\bullet$, as a graded complex over $((Z/\overline{A})_{\mathrm{qrsp}},\mathcal{O}_{\mathrm{qrsp}})$, is defined as the graded base change

$$\mathrm{Red}_{Z/A}(M^\bullet):=M^\bullet\widehat{\bigotimes}_{\mathrm{Gr}^\bullet_N\Delta^{(1)}_{-/A}}\mathcal{O}_{\mathrm{qrsp}}.$$

The i-th graded piece of $\mathrm{Red}_{Z/A}(M^\bullet)$ is denoted as $\mathrm{Red}_{i,Z/A}(M^\bullet)$.

(2) The graded complex $M^\bullet$ is said to *have weights in* $[a,b]$ if $M^i=0$ for $i\ll 0$ and

$$\mathrm{Red}_{i,Z/A}(M^\bullet)=0,\forall i\notin[a,b].$$

For a relative (F-)gauge $E=(E,\mathrm{Fil}^\bullet E^{(1)},\varphi_E)$ over Z/A we define its reduction and weights by that of its associated graded $M^\bullet=\mathrm{Gr}^\bullet E^{(1)}$.

Proposition 4.17. *Let Z be a quasi-syntomic $\overline{A}$-formal scheme, let $*\in\{\emptyset,\mathrm{perf},\mathrm{vect}\}$, and let $a\le b$ be two integers.*

(i) Assume $M^\bullet\in\mathcal{DG}^*_{p\text{-comp}}(Z_{\mathrm{qrsp}},\mathrm{Gr}^\bullet_N\Delta^{(1)}_{-/A})$ has weights in $[a,b]$. Then there is a unique increasing filtration $\mathrm{Fil}^{\mathrm{wt}}_i(M^\bullet)$ on $M^\bullet$ satisfying the following axioms:

(a) The filtration is exhaustive;

(b) the filtration "starts at 0": i.e. $\mathrm{Fil}^{\mathrm{wt}}_{\ll 0}(M^\bullet)=0$; and

(c) the graded piece $\mathrm{Gr}^{\mathrm{wt}}_i(M^\bullet)\simeq N_i\widehat{\otimes}_{\mathcal{O}_Z}\mathrm{Gr}^\bullet_N\Delta^{(1)}_{-/A}$ where $N_i\in\mathcal{DG}^*_{p\text{-comp}}(\mathcal{O}_Z)$ has grading i.

Moreover, the induced filtration $\mathrm{Fil}^{\mathrm{wt}}_i(\mathrm{Red}_{Z/A}(M^\bullet)):=\mathrm{Red}_{Z/A}(\mathrm{Fil}^{\mathrm{wt}}_i(M^\bullet))$ is the one induced by grading:

$$\mathrm{Fil}^{\mathrm{wt}}_i(\mathrm{Red}_{Z/A}(M^\bullet))\cong\bigoplus_{j\le i}\mathrm{Red}_{j,Z/A}(M^\bullet).$$

In particular, there are natural isomorphisms:

$$N_i \cong \mathrm{Red}_{Z/A}(\mathrm{Gr}_i^{\mathrm{wt}}(M^\bullet)) \cong \mathrm{Gr}_i^{\mathrm{wt}}(\mathrm{Red}_{Z/A}(M^\bullet)) \cong \mathrm{Red}_{i,Z/A}(M^\bullet)$$

in $\mathcal{DG}^*_{p\text{-}comp}(\mathcal{O}_Z)$. *Therefore, the filtration is indexed by* $i \in [a,b]$.

(ii) *Assume there is a map of* p*-adic formal schemes* $Z \to X$ *such that* X *is quasi-syntomic. Let* $E = (E, \mathrm{Fil}^\bullet E) \in \mathrm{Gauge}^*(X)$ *having weights in* $[a,b]$*, and let* E' *be the associated relative gauge over* Z/A *as in Theorem 4.13. Then* E' *also has weights in* $[a,b]$*. In fact we have a natural graded tensor product formula:*

$$\mathrm{Red}_{Z/A}(E') \cong \mathrm{Red}_X(E)\widehat{\otimes}_{\mathcal{O}_X}\mathcal{O}_Z.$$

Moreover the weight filtrations on $M^\bullet := \mathrm{Gr}^\bullet(E)$ *and* $N^\bullet := \mathrm{Gr}^\bullet(E')$ *are related via a natural filtered tensor product formula:*

$$\mathrm{Fil}_i^{\mathrm{wt}}(N^\bullet) \cong \mathrm{Fil}_i^{\mathrm{wt}}(M^\bullet)\widehat{\otimes}_{\mathrm{Gr}_N^\bullet}\mathrm{Gr}_N^\bullet{\mathbb{\Delta}}^{(1)}_{-/A}.$$

Proof. Part (i) is identical with the proof of Theorem 3.45 and we do not repeat here. For part (ii), it suffices to check the claim quasi-syntomic locally, and we assume there is a quasiregular semiperfectoid algebra $S \in X_{\mathrm{qrsp}}$, a large quasi-syntomic algebra $S' \in (Z/\overline{A})_{\mathrm{qrsp}}$, together with a map $S \to S'$ that is compatible with $X \to Z$. Then by proof of Theorem 4.13, we have $\mathrm{Fil}^\bullet E'(S')^{(1)} = \mathrm{Fil}^\bullet E(S)\widehat{\otimes}_{\mathrm{Fil}_N^\bullet{\mathbb{\Delta}}_S}\mathrm{Fil}_N^\bullet{\mathbb{\Delta}}^{(1)}_{S'/A}$. Notice that we also have a commutative diagram of graded rings

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[d] & \mathrm{Gr}_N^\bullet{\mathbb{\Delta}}_S \arrow[r, "\mathrm{Gr}^0"] \arrow[d] & S \arrow[d] \\ S' \arrow[r, "\iota"] & \mathrm{Gr}_N^\bullet{\mathbb{\Delta}}^{(1)}_{S'/A} \arrow[r, "\mathrm{Gr}^0"] & S'. \end{tikzcd}$$

As a consequence, by the local formula of the reduction functors (Construction 3.42, Definition 4.16), we get the tensor product formula

$$\mathrm{Red}_{S'/A}(E') \simeq \mathrm{Red}_S(E)\widehat{\otimes}_S S',$$

and the last formula follows from uniqueness of the weight filtration on $N^\bullet$. ☐

4.3. **Filtered Higgs complex and Hodge–Tate realization.** In eye of Proposition 4.4, we study the mod t specialization of a relative prismatic F-gauge in this subsection.

We start by considering the notion of filtered Higgs complex and the induced conjugate filtration on Hodge–Tate cohomology. Let (A, I) be a bounded prism. Recall that for a large quasi-syntomic formal scheme S over $\overline{A}$, the relative Hodge–Tate cohomology ring $\overline{\mathbb{\Delta}}_{S/A}$ admits an increasing filtration called *conjugate filtration*, whose i-th graded factor is $\wedge^i\mathbb{L}_{S/\overline{A}}\{-i\}[-i]$ (see [BS22, Construction 7.6]). We use $\mathrm{Gr}_\bullet^{\mathrm{conj}}\overline{\mathbb{\Delta}}_{S/A} = \bigoplus_{i\in\mathbb{N}}\wedge^i\mathbb{L}_{S/\overline{A}}\{-i\}[-i]$ to denote the associated graded algebra over S.

Definition 4.18. Let Z be a quasi-syntomic $\overline{A}$-formal scheme.

(i) The category of *filtered Higgs fields* over Z/A is defined as the limit of ∞-categories

$$\mathrm{FilHiggs}(Z/A) := \lim_{S\in(Z/\overline{A})_{\mathrm{qrsp}}}\mathrm{FilHiggs}(\mathrm{Spf}(S)/A),$$

where $\mathrm{FilHiggs}(\mathrm{Spf}(S)/A) := \mathcal{DF}_{p\text{-comp}}(\mathrm{Fil}_\bullet^{\mathrm{conj}}\overline{\mathbb{\Delta}}_{S/A})$ is the category of p-complete filtered complexes over the filtered ring $\mathrm{Fil}_\bullet^{\mathrm{conj}}\overline{\mathbb{\Delta}}_{S/A}$ for any large quasi-syntomic $\overline{A}$-formal scheme.

(ii) The category of *graded Higgs fields* over Z/A is defined as the limit of ∞-categories

$$\mathrm{GrHiggs}(Z/A) := \lim_{S\in(Z/\overline{A})_{\mathrm{qrsp}}}\mathrm{GrHiggs}(\mathrm{Spf}(S)/A),$$

where $\mathrm{GrHiggs}(\mathrm{Spf}(S)/A)$ is the category of p-complete graded complexes over the graded ring $\mathrm{Gr}_\bullet^{\mathrm{conj}}\overline{\mathbb{\Delta}}_{S/A}$ for any large quasi-syntomic $\overline{A}$-formal scheme.

Here, similar to Remark 3.24, for a map of large quasi-syntomic $\overline{A}$-algebras $S_1 \to S_2$, the completed filtered/graded base change induces natural functors between $\mathrm{FilHiggs}(\mathrm{Spf}(S_i)/A)$ and $\mathrm{GrHiggs}(\mathrm{Spf}(S_i)/A)$.

Similar to Definition 3.21, for $* \in \{\mathrm{vect}, \mathrm{perf}\}$, we use $\mathrm{FilHiggs}^*(S/A)$ and $\mathrm{GrHiggs}^*(S/A)$ to denote the corresponding subcategory spanned by p-completely finite projective or p-completely perfect objects. Then we may define $\mathrm{FilHiggs}^*(Z/A)$ and $\mathrm{GrHiggs}^*(Z/A)$ by similar limit formulae.

The analog of Proposition 3.23 still holds true:

Proposition 4.19. *Let S be a large quasi-syntomic $\overline{A}$-formal scheme and let Z be a quasi-syntomic $\overline{A}$-formal scheme. Then for $* \in \{\mathrm{vect}, \mathrm{perf}\}$, we may drop "p-completely", when defining* $\mathrm{FilHiggs}^*(S/A)$, $\mathrm{FilHiggs}^*(Z/A)$, $\mathrm{GrHiggs}^*(S/A)$ *and* $\mathrm{GrHiggs}^*(Z/A)$, *without changing the definition.*

Proof. The case of general Z follows from the case of large quasi-syntomic $\overline{A}$-algebras S. In this case, we are reduced to Corollary 2.27. □

Construction 4.20 (Associated graded functor)**.** There is a natural functor by taking associated graded:

$$\mathrm{Gr} : \mathrm{FilHiggs}^*(Z/A) \longrightarrow \mathrm{GrHiggs}^*(Z/A),$$

for $* = \{\emptyset, \mathrm{vect}, \mathrm{perf}\}$. When $Z = \mathrm{Spf}(S)$ is large quasi-syntomic, it sends a filtered $\mathrm{Fil}^{\mathrm{conj}}_\bullet \overline{\Delta}_{S/A}$-complex $\mathrm{Fil}_\bullet M$ to the graded $\mathrm{Gr}^{\mathrm{conj}}_\bullet \overline{\Delta}_{S/A}$-complex $\mathrm{Gr}_\bullet M = \bigoplus_{i\in\mathbb{N}} \mathrm{Fil}_i M/\mathrm{Fil}_{i-1} M$.

We also have the quasi-syntomic sheaf property for categories of filtered and graded Higgs complexes.

Proposition 4.21. *Let $S \to S^{(0)}$ be a quasi-syntomic cover of large quasi-syntomic rings over $\overline{A}$, and let $S^{(\bullet)}$ be the p-completed Čech nerve. Then for $* \in \{\emptyset, \mathrm{perf}, \mathrm{vect}\}$, we have a natural equivalence*

$$\mathrm{FilHiggs}^*(\mathrm{Spf}(S)/A) \simeq \lim_{[n]\in\Delta} \mathrm{FilHiggs}^*(\mathrm{Spf}(S^{(n)})/A),$$
$$\mathrm{GrHiggs}^*(\mathrm{Spf}(S)/A) \simeq \lim_{[n]\in\Delta} \mathrm{GrHiggs}^*(\mathrm{Spf}(S^{(n)})/A).$$

Proof. One just mimics the proof of Proposition 3.30, and uses Proposition 4.5 (2) as the main ingredient. □

Remark 4.22. Let us justify the appearance of "Higgs" in the above discussion. By construction, there is a forgetful functor from $\mathrm{FilHiggs}(Z/A)$ to the category of Hodge–Tate crystals over $(Z/A)_{\Delta}$, by forgetting the filtration structure. Moreover, when Z is affine and smooth over $\overline{A}$, showing in [BL22b, Corollary 6.6] (independently by [Tia21, Thm. 4.12], and for affine smooth Z that is small by [MW22, Thm. 1.1]) Hodge–Tate crystals over $Z/\overline{A}$ are equivalent to quasi-nilpotent Higgs fields in complexes of Z/A. So we can think of objects in $\mathrm{FilHiggs}(Z/A)$ as derived filtered generalizations of the usual notion of Higgs fields, over p-adic formal schemes that may not be smooth.

To relate the relative prismatic (F-)gauge with the filtered Higgs field, we first recall the following observation on Nygaard graded pieces of relative prismatic cohomology from [BS22, §15]. Assume the ideal I is generated by an element d, which we fix for the rest of this subsection. Then one can define a twisted version of the conjugate filtered Hodge–Tate cohomology, by

$$S \longmapsto \left(\mathrm{colim}\ \mathrm{Fil}^{\mathrm{conj}}_\bullet \overline{\Delta}_{S/A}\{\bullet\}\right)^\wedge_p,$$

where the transition map is given by the tensor product of the canonical map $\mathrm{Fil}^{\mathrm{conj}}_i \overline{\Delta}_{S/A} \to \mathrm{Fil}^{\mathrm{conj}}_{i+1} \overline{\Delta}_{S/A}$ with the twisting map $d : I^i/I^{i+1} \simeq I^{i+1}/I^{i+2}$.

Fact 4.23. *Let d be a generator of the ideal I. For a large quasi-syntomic $\overline{A}$-algebra S, the relative Frobenius $\varphi_{rel} \colon \Delta^{(1)}_{S/A} \to \Delta_{S/A}$ identifies the graded ring $\mathrm{Gr}^\bullet_N \Delta^{(1)}_{S/A}$ with the Rees construction of the twisted conjugate filtered Hodge–Tate cohomology $\mathrm{Fil}^{\mathrm{conj}}_\bullet \overline{\Delta}_{S/A}\{\bullet\}$. As a consequence, for $* \in \{\emptyset, \mathrm{perf}, \mathrm{vect}\}$, we have an equivalence*

$$\mathcal{DG}^*_{p\text{-comp}}(\mathrm{Gr}^\bullet_N \Delta^{(1)}_{S/A}) \simeq \mathcal{DF}^*_{p\text{-comp}}(\mathrm{Fil}^{\mathrm{conj}}_\bullet \overline{\Delta}_{S/A}).$$

Proof. The first identification of rings is in [BS22, Thm. 15.2.(2)] and also recorded in Proposition 4.4. To get the equivalence of categories, we simply notice that there is an equivalence of filtered derived categories

$$\begin{aligned}\mathcal{DF}(\mathrm{Fil}^{\mathrm{conj}}_{\bullet}\overline{\Delta}_{S/A}) &\longrightarrow \mathcal{DF}(\mathrm{Fil}^{\mathrm{conj}}_{\bullet}\overline{\Delta}_{S/A}\{\bullet\}),\\ \mathrm{Fil}_{\bullet}M &\longmapsto \mathrm{Fil}_{\bullet}M\otimes_{\overline{A}}\bar{I}^{\bullet}/\bar{I}^{\bullet+1},\end{aligned}$$

which is compatible with the twisting of the conjugate filtered Hodge–Tate cohomology. □

By taking limit ranging over all $S\in(Z/\overline{A})_{\mathrm{qrsp}}$, the above naturally extends to general quasi-syntomic $\overline{A}$-formal schemes.

Corollary 4.24. *Let Z be a quasi-syntomic $\overline{A}$-formal scheme, and let $*\in\{\emptyset,\mathrm{perf},\mathrm{vect}\}$. There is an equivalence of categories*

$$\mathcal{DG}^{*}_{p\text{-comp}}((Z/\overline{A}))_{\mathrm{qrsp}},\mathrm{Gr}^{\bullet}_{N}\Delta^{(1)}_{-/A})\simeq \mathrm{FilHiggs}^{*}(Z/A).$$

Definition 4.25. Let Z be a quasi-syntomic $\overline{A}$-formal scheme, and let $*\in\{\emptyset,\mathrm{perf},\mathrm{vect}\}$. We define the *Hodge–Tate realization* functor

$$\text{(F-)Gauge}^{*}(Z/A)\longrightarrow \mathrm{FilHiggs}^{*}(Z/A)$$

to be the composition of the associated graded functor with the equivalence in Corollary 4.24. Namely it is the limit of compositions

$$\text{(F-)Gauge}^{*}(\mathrm{Spf}(S)/A)\longrightarrow \mathcal{DG}^{*}_{p\text{-comp}}(\mathrm{Gr}^{\bullet}_{N}\Delta^{(1)}_{S/A})\simeq \mathcal{DF}^{*}_{p\text{-comp}}(\mathrm{Fil}^{\mathrm{conj}}_{\bullet}\overline{\Delta}_{S/A})=\mathrm{FilHiggs}^{*}(\mathrm{Spf}(S)/\overline{A}),$$

where S ranges over all $(Z/\overline{A})_{\mathrm{qrsp}}$.

The following relates Hodge–Tate cohomology and Higgs cohomology.

Proposition 4.26. *Let Z be a quasi-syntomic formal scheme over $\overline{A}$, and let $*\in\{\emptyset,\mathrm{perf},\mathrm{vect}\}$. Let $E=(E,\mathrm{Fil}^{\bullet}E^{(1)},\varphi_E)\in\text{F-Gauge}^{*}(Z/A)$ and let $(M,\mathrm{Fil}_{\bullet}M)$ be the associated filtered Higgs field. Then φ_E induces the following isomorphisms*

1. *twisted graded pieces $\mathrm{Gr}^{i}E^{(1)}\{-i\}$ of $\mathrm{Fil}^{\bullet}E^{(1)}$ and filtrations $\mathrm{Fil}_{i}M$ of M;*
2. *Hodge–Tate cohomology of E and Higgs cohomology of M.*

As a consequence, for a relative F-gauge E over X/A, we get a natural filtration on $R\Gamma((X/A)_{\Delta},\overline{E})$, which we call the *conjugate filtration* on Hodge–Tate cohomology of E.

Proof of Proposition 4.26. Let $E=(E,\mathrm{Fil}^{\bullet}E^{(1)},\varphi_E)$ be an object in $\text{F-Gauge}^{*}(\mathrm{Spf}(S)/A)$ for a given $S\in(Z/\overline{A})_{\mathrm{qrsp}}$, and let $(M,\mathrm{Fil}_{\bullet}M)$ be the associated filtered Higgs field. On the one hand, by Remark 4.7 the map φ_E induces an isomorphism of complexes

$$\Big(\mathrm{colim}_{i\in\mathbb{Z}}(\cdots\to \mathrm{Fil}^{i}E^{(1)}\otimes I^{-i}\to \mathrm{Fil}^{i+1}E^{(1)}\otimes I^{-i-1}\to\cdots)\Big)^{\wedge}_{(p,I)}\simeq E,$$

whose reduction mod I is the Hodge–Tate cohomology, namely

$$\Big(A/I\otimes_A \mathrm{colim}_{i\in\mathbb{Z}}(\cdots\to \mathrm{Fil}^{i}E^{(1)}\otimes I^{-i}\to \mathrm{Fil}^{i+1}E^{(1)}\otimes I^{-i-1}\to\cdots)\Big)^{\wedge}_{p}\simeq \overline{E}.$$

Moreover, by taking the mod I (where I is having filtration degree 1) reduction within the colimit, we can rewrite the left hand side as

$$\Big(\mathrm{colim}_{i\in\mathbb{Z}}(\cdots\to \mathrm{Gr}^{i}E^{(1)}\otimes \bar{I}^{-i}\to \mathrm{Gr}^{i+1}E^{(1)}\otimes \bar{I}^{-i-1}\to\cdots)\Big)^{\wedge}_{p}\simeq \overline{E}.$$

On the other hand, the twisted graded factor $\mathrm{Gr}^{\bullet}E^{(1)}\{-\bullet\}$ is the Rees construction for the associated filtered Higgs field $\mathrm{Fil}_{\bullet}M$. In particular, the Higgs cohomology, which is the underlying complex M of the increasingly filtered objects $\mathrm{Fil}_{i}M$, is equal to

$$(\mathrm{colim}_{i\in\mathbb{Z}}(\cdots\to \mathrm{Fil}_{i}M\to \mathrm{Fil}_{i+1}M\to\cdots))^{\wedge}_{p}\simeq \Big(\mathrm{colim}_{i\in\mathbb{Z}}(\cdots\to \mathrm{Gr}^{i}E^{(1)}\{-i\}\to \mathrm{Gr}^{i+1}E^{(1)}\{-i-1\}\to\cdots)\Big)^{\wedge}_{p}.$$

Hence the relative Frobenius identifies the Hodge–Tate cohomology of the F-gauge E with the Higgs cohomology of the associated filtered Higgs field M, and sends $\mathrm{Gr}^i E^{(1)}\{-i\}$ isomorphically onto $\mathrm{Fil}_i M$. For arbitrary quasi-syntomic $\overline{A}$-formal scheme Z and a relative F-gauge $E \in \text{F-Gauge}^*(Z/A)$, since the above isomorphisms for $E(S)$ is functorial in $S \in (Z/\overline{A})_{\mathrm{qrsp}}$, by passing to limit we may conclude the isomorphism of cohomology in general. □

The identification in Corollary 4.24 in particular allows us to study the notion of weights on filtered Higgs complexes. Following Definition 4.16, for $* \in \{\emptyset, \mathrm{perf}, \mathrm{vect}\}$, we let $\mathrm{Red}_{Z/A}$ be the base change functor

$$\mathrm{FilHiggs}^*(Z/\overline{A}) \xrightarrow{-\otimes_{\mathrm{Fil}^{\mathrm{conj}}_\bullet \overline{\Delta}_{-/A}} \mathcal{O}_{\mathrm{qrsp}}} \mathcal{DG}^*_{p\text{-comp}}(\mathcal{O}_Z),$$

which by construction factors through the quotient functor $\mathrm{Gr}\colon \mathrm{FilHiggs}^*(Z/\overline{A}) \to \mathrm{GrHiggs}^*(Z/\overline{A})$. A filtered Higgs complex M is of *weight* $[a,b]$ if $\mathrm{Red}_{Z/A}(M)$ lives within degree $[a,b]$.

Translating Proposition 4.17 using Corollary 4.24, we get the weight filtration on filtered Higgs fields.

Corollary 4.27. *Fix a generator d of the ideal I, let Z be a quasi-syntomic p-adic formal scheme over $\overline{A}$. Let $* \in \{\emptyset, \mathrm{perf}, \mathrm{vect}\}$, and let $a \le b$ be two integers. For a filtered Higgs complex $M \in \mathrm{FilHiggs}^*(Z/\overline{A})$ of weight $[a,b]$, there is a unique increasing filtration $\mathrm{Fil}^{\mathrm{wt}}_i M$ on M satisfying the following axioms:*

(1) The filtration is exhaustive;
(2) the filtration "starts at 0": i.e. $\mathrm{Fil}^{\mathrm{wt}}_{\ll 0} M = 0$; and
(3) the graded piece $\mathrm{Gr}^{\mathrm{wt}}_i M \simeq N_i \otimes_{\mathcal{O}_Z} \mathrm{Fil}^{\mathrm{conj}}_\bullet \overline{\Delta}_{-/A}$ where $N_i \in \mathcal{DG}^*_{p\text{-comp}}(\mathcal{O}_Z)$ has grading i.

Moreover, the induced filtration $\mathrm{Fil}^{\mathrm{wt}}_i(\mathrm{Red}_{Z/A}(M)) \coloneqq \mathrm{Red}_{Z/A}(\mathrm{Fil}^{\mathrm{wt}}_i M)$ is the one induced by grading:

$$\mathrm{Fil}^{\mathrm{wt}}_i(\mathrm{Red}_{Z/A}(M)) \cong \bigoplus_{j\le i} \mathrm{Red}_{j,Z/A}(M).$$

In particular, there are natural isomorphisms:

$$N_i \cong \mathrm{Red}_{Z/A}(\mathrm{Gr}^{\mathrm{wt}}_i(M^\bullet)) \cong \mathrm{Gr}^{\mathrm{wt}}_i(\mathrm{Red}_{Z/A}(M)) \cong \mathrm{Red}_{i,Z/A}(M)$$

in $\mathcal{DG}^*_{p\text{-comp}}(\mathcal{O}_Z)$. Therefore, the filtration is indexed by $i \in [a,b]$.

By taking the direct image to the Zariski site of Z, we can analyze the graded factors of the conjugate filtration on Hodge–Tate cohomology.

Construction 4.28. Denote by $\lambda\colon Z_{\mathrm{qrsp}} \to Z_{\mathrm{Zar}}$ the natural map of ringed sites. Then there is a natural graded derived pushforward functor between two graded derived category of presheaves

$$R\lambda_*\colon \mathcal{DG}((Z/\overline{A})_{\mathrm{qrsp}}, \mathcal{O}^{\mathrm{PSh}}_{\mathrm{qrsp}}) \to \mathcal{DG}(Z_{\mathrm{Zar}}, \mathcal{O}^{\mathrm{PSh}}_Z).$$

Remark 4.29. For any open $U \subset Z$, the evaluation of $R\lambda_*$ at U of quasi-syntomic sheaves is given by the unfolding process, see Proposition 4.1 and Remark 4.2.

We now use weight filtration to subdivide the Hodge–Tate cohomology and Higgs cohomology into smaller pieces, where each of them can be understood using the reduction functor and differential forms.

Corollary 4.30. *Let Z be a smooth $\overline{A}$-formal scheme of relative dimension n, let $* \in \{\mathrm{vect}, \mathrm{perf}\}$, and let $M \in \mathrm{FilHiggs}^*(Z/\overline{A})$ be of weight $[a,b]$. The j-th graded factor $\mathrm{Gr}_j R\lambda_* M$ is zero unless $j \in [a, b+n]$. In the latter case it admits a finite increasing filtration such that the i-th graded factor of this filtration is*

$$(\mathrm{Red}_{a+i,Z/A}(M)) \otimes_{\mathcal{O}_X} \Omega^{j-a-i}_{Z/\overline{A}}\{a+i-j\}[a+i-j], \quad \max\{0, j-a-n\} \le i \le \min\{j,b\} - a.$$

In particular, if $\bigoplus_l \mathrm{Red}_{l,Z/A}(M)$ is given by a coherent sheaf over $\mathcal{O}_Z$, then we have

$$\mathrm{Gr}_j R\lambda_* M \in D^{[\max\{j-b,0\}, \min\{j-a,n\}]}_{\mathrm{coh}}(\mathcal{O}_Z).$$

Here in the calculation, we implicitly use the fact that for a smooth $\overline{A}$-formal scheme of relative dimension n, the relative sheaf of differential $\Omega^l_{Z/\overline{A}}$ is zero unless $l \in [0, n]$.

For the future usage, we also record a special case when an F-gauge comes from an I-torsionfree coherent F-crystal, where we can show that the reduction of the associated filtered Higgs field (which is the same as the reduction of the gauge by Corollary 4.24) is coherent. This is summarized in the next result.

Corollary 4.31. *Let $f : X \to Y$ be a smooth morphism of smooth p-adic formal schemes over $\mathcal{O}_K$, and let $(\mathcal{E}, \varphi_\mathcal{E})$ be an I-torsionfree coherent F-crystal over X. For a bounded prism $(A, I) \in Y_{\Delta}$ such that $\mathrm{Spf}(\overline{A}) \to Y$ is p-completely flat, let $E' = BC_{X,X_{\overline{A}}/A}(\Pi_X(\mathcal{E}))$ be the associated relative F-gauge over $X_{\overline{A}}/A$ via Theorem 3.32 and Theorem 4.13. Then the reduction $\mathrm{Red}_{X_{\overline{A}}/A}(E')$ is a graded coherent sheaf over $\mathcal{O}_X$.*

Proof. Using the identification in Corollary 4.24, this follows from the complete base change formula $\mathrm{Red}_{X_{\overline{A}}/A}(E') \simeq \mathrm{Red}_X(E)\widehat{\otimes}_{\mathcal{O}_X}\mathcal{O}_{X_{\overline{A}}}$ in Proposition 4.17.(ii): since $\mathrm{Red}_X(E)$ is coherent over $\mathcal{O}_X$ (Theorem 3.47) and $X_{\overline{A}} \to X$ is p-completely flat, the complete base change is also coherent. □

4.4. **Completeness of Nygaard filtration via filtered de Rham realization.** In this subsection, in eye of Proposition 4.4, we study the mod $\frac{d}{t}$ specialization of a relative prismatic F-gauge in this subsection. The following definition and sheaf-property proposition is completely analogous to previous subsections.

Definition 4.32. Let Z be a quasi-syntomic $\overline{A}$-formal scheme. The category of *filtered connections* over $Z/\overline{A}$ is defined as the limit of ∞-categories

$$\mathrm{FilConn}(Z/\overline{A}) := \lim_{S \in (Z/\overline{A})_{\mathrm{qrsp}}} \mathrm{FilConn}(\mathrm{Spf}(S)/\overline{A}),$$

where $\mathrm{FilConn}(\mathrm{Spf}(S)/\overline{A}) := \mathcal{DF}_{p\text{-comp}}(\mathrm{Fil}^\bullet_H \mathrm{dR}(S/\overline{A}))$ is the category of p-complete filtered complexes over the filtered ring $\mathrm{Fil}^\bullet_H \mathrm{dR}(S/\overline{A})$. Here, similar to Remark 3.24, for a map of large quasi-syntomic $\overline{A}$-algebras $S_1 \to S_2$, the completed filtered base change induces natural functors between $\mathrm{FilConn}(\mathrm{Spf}(S_i)/\overline{A})$.

Similar to Definition 3.21, for $* \in \{\mathrm{vect}, \mathrm{perf}\}$, we use $\mathrm{FilConn}^*(Z/\overline{A})$ to denote the subcategory spanned by p-completely finite projective (resp. perfect) objects.

The analog of Proposition 3.23 still holds true:

Proposition 4.33. *Let S be a large quasi-syntomic $\overline{A}$-formal scheme and let Z be a quasi-syntomic $\overline{A}$-formal scheme. Then for $* \in \{\mathrm{vect}, \mathrm{perf}\}$, we may drop "p-completely", when defining $\mathrm{FilConn}^*(S/\overline{A})$ and $\mathrm{FilConn}^*(Z/\overline{A})$, without changing the definition.*

Proof. The case of $\mathrm{FilConn}^*(Z/\overline{A})$ follows from the case of $\mathrm{FilConn}^*(S/\overline{A})$ by descent. The proof of $\mathrm{FilConn}^*(S/\overline{A})$ follows from the exact proof of Proposition 4.9: Note that in the said proof, we have explained why $(\mathrm{dR}_{S/\overline{A}}, \mathrm{Fil}^1_H \mathrm{dR}(S/\overline{A}))$ is a henselian pair. □

Just like before, all these form quasi-syntomic sheaves.

Proposition 4.34. *Let $S \to S^{(0)}$ be a quasi-syntomic cover of large quasi-syntomic rings over $\overline{A}$, and let $S^{(\bullet)}$ be the p-completed Čech nerve. Then for $* \in \{\mathrm{perf}, \mathrm{vect}\}$, we have a natural equivalence*

$$\mathrm{FilConn}^*(\mathrm{Spf}(S)/\overline{A}) \simeq \lim_{[n]\in\Delta} \mathrm{FilConn}^*(\mathrm{Spf}(S^{(n)})/\overline{A}).$$

Proof. One just mimics the proof of Proposition 3.30, and uses Proposition 4.5 (3) as the main ingredient. □

In [BS22, Lemma 8.6] one finds an interesting example of (p, I)-completely descendable map (see Digression 2.39) between two (p, I)-complete $\mathbb{E}_\infty$-rings, what we need is the following slight generalization.

Proposition 4.35. *Let $(A, I = (d))$ be a bounded prism with $\overline{A} := A/I$, let $\overline{A}\langle \underline{X} \rangle \to R$ be a p-completely étale map, and let $R_\infty := R\widehat{\otimes}_{\overline{A}\langle \underline{X}\rangle}\overline{A}\langle \underline{X}^{1/p^\infty}\rangle$. Then the filtered map*

$$\mathrm{Fil}^\bullet_N \mathrm{R\Gamma}(\mathrm{Spf}(R)_{\mathrm{qSyn}}, \Delta^{(1)}_{-/A}) \to \mathrm{Fil}^\bullet_N \Delta^{(1)}_{R_\infty/A}$$

induces a (p, I)-completely descendable map between their underlying Rees algebras. Similarly, the filtered map

$$\mathrm{Fil}^\bullet_H \mathrm{dR}(R/\overline{A}) \to \mathrm{Fil}^\bullet_H \mathrm{dR}(R_\infty/\overline{A})$$

induces a p-completely descendable map between their underlying Rees algebras.

Proof. Since descendability is preserved under base change, we are immediately reduced to the case where $\overline{A}\langle \underline{X}\rangle = R$. It suffices to prove the statement when there is only one variable, as the general case follows from tensoring up one variable case. Just like the proof of [BS22, Lemma 8.6], we use F to denote the fiber in both Rees algebra maps, and we just need to show $F^{\otimes 2}$ has only nullhomotopic maps to the Rees algebra for the filtered sheaves evaluated at $\mathrm{Spf}(R)$.

For the case of Nygaard filtered $\Delta^{(1)}_{-/A}$, by $\frac{d}{t}$-completeness, it suffices to check the above claim after modulo $\frac{d}{t}$ (using notation from Proposition 4.4); by the Proposition 4.4 we are reduced to the case of Hodge filtered $\mathrm{dR}(-/\overline{A})$. At this point, we simply proceed as in the proof of [BS22, Lemma 8.6]: By completeness we may further reduce mod p, then our map has a model defined over $\mathbb{F}_p$, and the proof as in loc. cit. applies to show the relevant mapping space has only one component. $\square$

Combining all the above discussions, we get an equivalent definition of (F-)gauges and filtered connections as certain filtered complexes over the filtered cohomology rings. Roughly speaking, this is saying that the relative versions of the filtered prismatization $(X/A)^N$, the syntomication $(X/A)^{\mathrm{syn}}$ and the filtered de Rham stack $(X/\overline{A})^{\mathrm{dR}+}$ can be built easily from the $\mathbb{G}_m$-quotients of affine stacks.

Corollary 4.36. *Let $(A, I = (d))$ be a bounded oriented prism with $\overline{A} := A/I$, and let $X = \mathrm{Spf}(R)$ be a p-complete smooth affine formal scheme over $\overline{A}$. Let $* \in \{\emptyset, \mathrm{perf}\}$.*

1. *(1) The category $\mathrm{Gauge}^*(X/A)$ is equivalent to the category of pairs $(E, \mathrm{Fil}^\bullet E^{(1)})$, where $E \in D^*_{(p,I)\text{-}comp}(\Delta_{X/A})$ and $\mathrm{Fil}^\bullet E^{(1)} \in \mathcal{DF}^*_{(p,I)\text{-}comp}(\mathrm{Fil}^\bullet_N \Delta^{(1)}_{R/A})$ with underlying complex $E^{(1)} := E\widehat{\otimes}_{A,\varphi_A} A$.*
2. *(2) The category $\text{F-Gauge}^*(X/A)$ is equivalent to the category of triples $(E, \mathrm{Fil}^\bullet E^{(1)}, \widetilde{\varphi}_E)$, where $(E, \mathrm{Fil}^\bullet E^{(1)})$ is as in (1) and $\widetilde{\varphi}_E : \mathrm{Fil}^\bullet E^{(1)} \widehat{\otimes}_{\mathrm{Fil}^\bullet_N \Delta^{(1)}_{R/A}, \varphi} I^{\mathbb{Z}} \Delta_{X/A} \to E \otimes_{\Delta_{X/A}} I^{\mathbb{Z}} \Delta_{R/A}$ is a filtered isomorphism.*
3. *(3) The category $\mathrm{FilConn}^*(X/\overline{A})$ is equivalent to the category $\mathcal{DF}^*_{p\text{-}comp}(\mathrm{dR}(R/\overline{A}))$.*

Proof. This follows from combining Proposition 2.40 and Proposition 4.35. $\square$

Using the descendability, below we also show the completeness of the filtration of an (F-)gauge in perfect complexes over a smooth p-adic formal scheme.

Theorem 4.37. *Let Z be a smooth $\overline{A}$-formal scheme. Let $E = (E, \mathrm{Fil}^\bullet E^{(1)})$ be a prismatic gauge in perfect complexes over Z/A, let $\mathcal{E}$ be a prismatic crystal in perfect complexes over Z/A, and let $(\mathcal{F}, \mathrm{Fil}^\bullet \mathcal{F})$ be a filtered perfect complex with connection over $Z/\overline{A}$.*

1. *(1) For any affine open $U = \mathrm{Spf}(R) \subset Z$ which admits a formal étale map to $\widehat{\mathbb{A}^n}_{\overline{A}}$, the filtered complexes $\mathrm{Fil}^\bullet E^{(1)}(U)$ and $\mathrm{Fil}^\bullet \mathcal{F}(U)$ are (p, I)-completely and p-completely perfect as filtered $\mathrm{Fil}^\bullet_N \Delta^{(1)}_{U/A}$-complexes and $\mathrm{Fil}^\bullet_H \mathrm{dR}(U/\overline{A})$-complexes respectively.*
2. *(2) The assignments $S \mapsto \mathcal{E}(\Delta_{S/A})$ and $S \mapsto \mathcal{E}^{(1)}(\Delta_{S/A}) := \mathcal{E}(\Delta_{S/A})\widehat{\otimes}_{A,\varphi_A} A$ defines quasi-syntomic sheaves on $(Z/A)_{\mathrm{qrsp}}$. With notation as in (1), the complex $\mathcal{E}(U)$ is (p, I)-completely perfect over $\Delta_{U/A}$. Moreover the natural map*
$$\mathcal{E}(U)\widehat{\otimes}_{A,\varphi_A} A \to \mathcal{E}^{(1)}(U)$$
is an isomorphism.
3. *(3) The filtrations $\mathrm{Fil}^\bullet E^{(1)}(Z)$ and $\mathrm{Fil}^\bullet \mathcal{F}(Z)$ are complete.*

Here the evaluation at U of quasi-syntomic sheaves is defined by the unfolding process, see Proposition 4.1 and Remark 4.2.

Proof. For (1): Fix a p-completely étale map $\overline{A}\langle \underline{X}\rangle \to R$. Let $R^{(\bullet)}$ be the Cech nerve of $R \to R_\infty$ where $R_\infty := R\widehat{\otimes}_{\overline{A}\langle \underline{X}\rangle} \overline{A}\langle \underline{X}^{1/p^\infty}\rangle$. By Remark 4.2, the values $\mathrm{Fil}^\bullet E^{(1)}(U)$ and $\mathrm{Fil}^\bullet \mathcal{F}(U)$ are given by the descent of their values on $R^{(\bullet)}$. By Proposition 2.17 and Definition 2.19, we need to check their associated Rees's construction is perfect after derived reduction mod (p, I) and p respectively. Therefore we are reduced to Proposition 4.35, by descent of completely perfectness along completely descendable map (Digression 2.39).

For (2): the first statement follows from the fact that $\Delta_{-/A}$ and $\Delta^{(1)}_{-/A}$ are quasi-syntomic sheaves (by Hodge–Tate and de Rham comparison respectively), and the perfectness assumption on $\mathcal{E}$. Then the completely perfectness of $\mathcal{E}(U)$ over $\Delta_{U/A}$ can be proved exactly as in (1). Lastly, with notations as in the proof of (1), we need to show the following natural arrow:

$$\mathcal{E}(U)\widehat{\otimes}_{A,\varphi_A}A \to \lim_{\bullet\in\Delta}\mathcal{E}^{(1)}(R^{(\bullet)})$$

is an isomorphism. We first observe base change formulas: $\mathcal{E}(U)\widehat{\otimes}_{A,\varphi_A}A \cong \mathcal{E}(U)\widehat{\otimes}_{\Delta_{U/A}}\Delta^{(1)}_{U/A}$ as well as $\mathcal{E}^{(1)}(R^{(\bullet)}) \cong \mathcal{E}(U)\widehat{\otimes}_{\Delta_{U/A}}\Delta^{(1)}_{R^{(\bullet)}/A}$. By completely perfectness of $\mathcal{E}(U)$ over $\Delta_{U/A}$, using the above base change formulas, we are reduced to showing that the unfolding of $\Delta^{(1)}_{-/A}$ at U is $\Delta_{U/A}\widehat{\otimes}_{A,\varphi_A}A$: This follows from the de Rham comparison as in [BS22, Theorem 15.3].

As for (3): quasi-syntomic sheaves are in particular Zariski sheaves, so it suffices to check completeness for the values on a basis of opens in Z_{Zar}. Note that by derived Nakayama lemma, we are reduced to checking completeness after applying $-/^L(p,I)$ and $-/^Lp$ respectively to these values. To that end, we simply use (1) and Proposition 2.18 to reduce ourselves to the completeness of the Nygaard (resp. Hodge) filtration on the Frobenius-twisted prismatic cohomology (resp. derived de Rham cohomology) of smooth algebras. The Hodge filtration for smooth formal schemes is a finite filtration, hence complete. The claim about Nygaard filtration can be found in [LL20, Lemma 7.8.(1)], the proof is quite easy: by $\frac{d}{t}$-completeness, we may check the completeness of filtration after modulo $\frac{d}{t}$, but then Proposition 4.4 reduces us to checking completeness of Hodge filtration for smooth formal schemes. □

In a private communication, Bhatt told us that [BL22b, Theorem 7.17] is proved by a similar argument.

5. Height of prismatic cohomology

In this section, we study Verschiebung operator and the Frobenius height of prismatic cohomology, with coefficients in coherent prismatic F-crystals.

Recall that in Construction 4.28 we defined a natural graded derived pushforward functor between two graded derived category of presheaves

$$R\lambda_* : \mathcal{DG}((X/\overline{A})_{\mathrm{qrsp}}, \mathcal{O}^{\mathrm{PSh}}_{\mathrm{qrsp}}) \to \mathcal{DG}(X_{\mathrm{Zar}}, \mathcal{O}^{\mathrm{PSh}}_X).$$

Also recall that in Theorem 3.32 we have attached an F-gauge $\Pi_X(\mathcal{E})$ on X to an F-crystal $(\mathcal{E},\varphi_{\mathcal{E}})$ on X, then according to Theorem 4.13, we obtain a relative F-gauge $BC_{X,X_{\overline{A}}/A}(\Pi_X(\mathcal{E}))$ over $X_{\overline{A}}/A$. Using these constructions, our first main theorem is the following.

Theorem 5.1. *Let $f\colon X \to Y$ be a smooth morphism of smooth p-adic formal schemes over $\mathcal{O}_K$ that is of relative dimension n, and let (A,I) be a bounded prism over Y such that $\mathrm{Spf}(\overline{A}) \to Y$ is p-completely flat. Assume $(\mathcal{E},\varphi_{\mathcal{E}}) \in \mathrm{F\text{-}Crys}^{\mathrm{coh}}(X)$ is I-torsionfree of height $[a,b]$, with $(E,\mathrm{Fil}^\bullet E^{(1)},\widetilde{\varphi}_E)$ the associated F-gauge.*

(i) For each $j \in \mathbb{Z}$, the complex $R\lambda_*\mathrm{Gr}^j E^{(1)}$ lives in $D^{[0,n]}_{\mathrm{coh}}(\mathcal{O}_{X_{\overline{A}}})$.

(ii) The filtered complex $R\lambda_*\mathrm{Fil}^{\geq n+b}E^{(1)}$ is equivalent to the I-adic filtration $R\lambda_* I^{\geq n+b}E^{(1)} = I^{\geq n+b}\otimes R\lambda_*E^{(1)}$.

(iii) The map $\widetilde{\varphi}_E$ induces a natural isomorphism of the truncations

$$\varphi_{R^{\leq i}\lambda_*} : R^{\leq i}\lambda_*(\mathrm{Fil}^{i+b}E^{(1)}) \simeq R^{\leq i}\lambda_*(I^{i+b}E) = I^{i+b}\otimes R^{\leq i}\lambda_*E,\ \forall i \in \mathbb{N}.$$

Combine (i) and (ii) above, we see that $R\lambda_*\mathcal{E}$ and $R\lambda_*\mathrm{Fil}^\bullet E^{(1)}$ live in $D^{[0,n]}(X_{\overline{A}})$.

Remark 5.2. Note that in the special case when $\mathcal{E}$ is the prismatic structure sheaf $\mathcal{O}_{\Delta}$, this is shown in [LL20, Lem. 7.8], which essentially follows from the calculation of graded pieces of Nygaard filtration as in [BS22, Thm. 15.2].

Proof. Denote by $(\overline{\mathcal{E}},\mathrm{Fil}_\bullet\overline{\mathcal{E}})$ the associated filtered Higgs field as in Definition 4.25. By Corollary 4.31 and Corollary 4.30, for each $l \in \mathbb{Z}$, we have

$$R\lambda_*\mathrm{Gr}_l\overline{\mathcal{E}} \in D^{[0,n]}_{\mathrm{coh}}(\mathcal{O}_X).$$

Since $R\lambda_*\mathrm{Fil}_j\overline{\mathcal{E}}$ admits a finite increasing and exhaustive filtration with graded pieces being $R\lambda_*\mathrm{Gr}_l\overline{\mathcal{E}}$, we see the cohomological bound and coherence in (i) hold for $R\lambda_*\mathrm{Fil}_j\overline{\mathcal{E}}$.

To compare the two filtrations, by the filtered completeness of $\mathrm{Fil}^\bullet E^{(1)}$ in Theorem 4.37 and that of the I-adic filtration, it suffices to compare the map of associated graded pieces. We let C^j be the cone of the Frobenius map $\varphi_{\mathcal{E}} : \mathrm{Fil}^j E^{(1)} \to I^j E$, regarded as a descending filtered complex via the filtration $C^{\geq j}$. Then for each $j \in \mathbb{Z}$, the graded piece $\mathrm{Gr}^j C^\bullet$ by definition is

$$\mathrm{Cone}(\varphi_{\mathcal{E}} : \mathrm{Gr}^j E^{(1)} \longrightarrow \overline{\mathcal{E}}\{j\}),$$

with the left hand side identified with $\mathrm{Fil}_j\overline{\mathcal{E}}\{j\}$ by Proposition 4.26. (i). So we get $\mathrm{Gr}^j C^\bullet \simeq \mathrm{Fil}_{\geq j+1}\overline{\mathcal{E}}\{j\}$. By Corollary 4.31 and Corollary 4.30, the filtered complex $R\lambda_*(\mathrm{Fil}_{\geq j+1}\overline{\mathcal{E}}\{j\})$ has a finite increasing and exhaustive filtration ranging from the filtered degree $j+1-b$ to n (and the entire term vanishes when $j+1-b > n$.), such that the l-th graded piece lives in $D^{[\max\{l,0\},n]}_{\mathrm{coh}}(\mathcal{O}_{X_{\overline{A}}})$. So by induction and the aforementioned facts in Corollary 4.30 again, we have

$$R\lambda_*\mathrm{Gr}^j C^\bullet \in D^{[\max\{j+1-b,0\},n]}_{\mathrm{coh}}(\mathcal{O}_{X_{\overline{A}}}), \tag{$*$}$$

which vanishes when $j+1-b > n$. This means that the Frobenius map induces an isomorphism from $R\lambda_*\mathrm{Gr}^j E^{(1)}$ to $R\lambda_*\overline{\mathcal{E}}\{j\}$ for $j+1-b>n$, and $R\lambda_*(\mathrm{Fil}^\bullet E^{(1)})$ is eventually the I-adic filtration.

Finally by the completeness of the filtration, item (iii) follows from a reformulation of $(*)$ that for each $j \geq i+b$, we have $R\lambda_*\mathrm{Gr}^j C^\bullet$ lives in $D^{[i+1,n]}_{\mathrm{coh}}(\mathcal{O}_{X_{\overline{A}}})$. $\square$

By taking the higher direct image along a proper smooth morphism, we can estimate the height of each individual prismatic cohomology. This is analogous to the work of Kedlaya bounding slopes of rigid cohomology as in [Ked06, Thm. 6.7.1].

Theorem 5.3. *Let $f\colon X \to Y$ be a smooth morphism of smooth p-adic formal schemes over $\mathcal{O}_K$ that is of relative dimension n, and let $(A,I) \in Y_{\Delta}$ such that $\mathrm{Spf}(\overline{A}) \to Y$ is p-completely flat. Assume $(\mathcal{E},\varphi_{\mathcal{E}}) \in \mathrm{F}\text{-}\mathrm{Crys}^{\mathrm{coh}}(X)$ is I-torsionfree of height $[a,b]$, and we denote $E = (E, \mathrm{Fil}^\bullet E^{(1)}, \widetilde{\varphi}_E)$ the associated relative F-gauge over $X_{\overline{A}}/A$. For an integer $j \geq a$ and a non-negative integer i, the Frobenius structure on the i-th relative prismatic cohomology induces a natural commutative diagram*

$$\begin{tikzcd}
\mathrm{H}^i((X/\overline{A})_{\mathrm{qrsp}}, \mathrm{Fil}^j E^{(1)}) \arrow[r, "u^j"] \arrow[d, "\varphi^j"'] & \mathrm{H}^i((X/\overline{A})_{\mathrm{qrsp}}, E^{(1)}) \arrow[d, "\varphi"] \\
I^j \otimes \mathrm{H}^i((X/\overline{A})_{\mathrm{qrsp}}, E) \arrow[r] & I^a \otimes \mathrm{H}^i((X/\overline{A})_{\mathrm{qrsp}}, E),
\end{tikzcd}$$

where the horizontal arrows are defined by the canonical maps and the vertical arrows are given by Frobenius morphisms on E. Then we have:

(i) the map φ^j is an isomorphism when $j \geq b + \min\{i,n\}$, and
(ii) the map u^j is an isomorphism when $j \leq a + \max\{0, i-n\}$.

Proof. We first notice that as the F-crystal $(\mathcal{E},\varphi_{\mathcal{E}})$ is I-torsionfree and of height $[a,b]$, the Frobenius map sends $\mathcal{E}^{(1)}$ into $I^a\mathcal{E} \simeq I^a \otimes \mathcal{E} \subset \mathcal{E}[1/I]$, and sends $\mathrm{Fil}^\bullet E^{(1)}$ into $I^a \otimes E$. So the diagram in the statement can be obtained by applying the functor $\mathrm{H}^i(X,-)$ at the following diagram of complexes over $(X_{\overline{A}})_{\mathrm{Zar}}$:

$$\begin{tikzcd}
R\lambda_*\mathrm{Fil}^j E^{(1)} \arrow[rr, "v^j"] \arrow[d, "R\lambda_*\widetilde{\varphi}^j_E"'] & & R\lambda_* E^{(1)} \arrow[d, "R\lambda_*\widetilde{\varphi}_E"] \\
I^j \otimes R\lambda_* E \arrow[r] & R\lambda_*(I^a \otimes E) \simeq & I^a \otimes R\lambda_* E.
\end{tikzcd}$$

To show item (i), we apply $\mathrm{H}^i(X,-)$ at the isomorphism of sheaves from Theorem 5.1.(iii), which immediately implies that the map φ^j is an isomorphism when $j \geq b+i$. It is then left to show that the map φ^j is an isomorphism when $j \geq b+n$, and we assume now that $i \geq n$. Since both $R\lambda_*\mathrm{Fil}^\bullet E^{(1)}$ and $R\lambda_* E$

live in cohomological degree $[0,n]$ (Theorem 5.1.(i) and (ii)), the Leray spectral sequence for the map $\varphi^j = \mathrm{H}^i(X, R\lambda_*\widetilde{\varphi}_E^j)$ is computed by the following n-terms

$$\mathrm{H}^{i-n}(X, R^n\lambda_*\widetilde{\varphi}_E^j), \ldots, \mathrm{H}^i(X, R^0\lambda_*\widetilde{\varphi}_E^j).$$

Notice that by Theorem 5.1.(iii), the map $R^l\lambda_*\widetilde{\varphi}_E^j$ is an isomorphism when $j \geq b+l$. Thus the map φ^j is an isomorphism when $j \geq b+n$, as the number l ranges from 0 to n.

For (ii), the sheaf of complexes $\mathrm{Cone}(v^j)$ admits a decreasing filtration, which by Proposition 4.26 satisfies

$$\mathrm{Gr}^l(\mathrm{Cone}(v^j)) = \begin{cases} \mathrm{Fil}_l\bar{\mathcal{E}}\{l\}, \ l \leq j-1, \\ 0, \ l \geq j. \end{cases}$$

Moreover, by Corollary 4.31 and Corollary 4.30, each $\mathrm{Fil}_l\bar{\mathcal{E}}\{l\}$ lives in $D_{\mathrm{coh}}^{[0,\min\{l-a,n\}]}(\mathcal{O}_X)$. So by the filtered completeness of $\mathrm{Cone}(v^j)$, the complex $\mathrm{Cone}(v^j)$ itself lives in $D_{\mathrm{coh}}^{[0,\min\{j-1-a,n\}]}(\mathcal{O}_X)$, which is zero when $j-1-a<0$, or equivalently $j \leq a$. This in particular implies that when $j \leq a$, the map v^j and hence u^j are always isomorphisms for all the integers i. To finish the proof of (ii), we are left to consider the case when $i \geq n$. Using Leray spectral sequence again, the cone of the map $u^j = \mathrm{H}^i(v^j)$ is computed by the following terms

$$\mathrm{H}^{i-n}(X, \mathcal{H}^n(\mathrm{Cone}(v^j)), \ldots, \mathrm{H}^n(X, \mathcal{H}^{i-n}(\mathrm{Cone}(v^j)),$$

where we implicitly use the fact that $\mathrm{H}^l(X, \mathcal{H}^{i-l}(\mathrm{Cone}(v^j)) = 0$ for $l > n$. In particular, when $j \leq a+i-n$, as the complex $\mathrm{Cone}(v^j)$ lives in $D_{\mathrm{coh}}^{[0,\min\{j-1-a,n\}]}(\mathcal{O}_X) \subset D_{\mathrm{coh}}^{[0,\min\{i-n-1,n\}]}(\mathcal{O}_X)$, the terms computing $\mathrm{Cone}(u^j)$ above are all equal to zero. Hence the map u^j is an isomorphism when $j \leq a+i-n$, under the assumption that $i \geq n$. □

Analogous to [BS22, Cor. 15.5], we can extend the Verschiebung operator on individual prismatic cohomology sheaf to general coefficients.

Corollary 5.4. *Keep the same assumptions and notations as in Theorem 5.1. Then for $i \in \mathbb{N}$, there is a natural map*

$$\psi : I^{i+b} \otimes_A R^{\leq i}\lambda_*E \longrightarrow R^{\leq i}\lambda_*E^{(1)},$$

such that its compositions with Frobenius morphism $\varphi : R^{\leq i}\lambda_*E^{(1)} \to R^{\leq i}\lambda_*E$ are canonical maps induced from inclusions $I^{i+b} \subset A$, namely

$$\varphi \circ \psi \simeq \mathrm{can} : I^{i+b} \otimes R^{\leq i}\lambda_*E \longrightarrow R^{\leq i}\lambda_*E;$$
$$\psi \circ \varphi \simeq \mathrm{can} : I^{i+b} \otimes R^{\leq i}\lambda_*E^{(1)} \longrightarrow R^{\leq i}\lambda_*E^{(1)}.$$

Proof. Let C be the sheaf of complexes $R^{\leq i}\lambda_*E$, and let $C^{(1)}$ be its φ_A-linear twist, which by flatness is $R^{\leq i}\lambda_*E^{(1)}$. Similarly we let $\mathrm{Fil}^jC^{(1)}$ be the complex $R^{\leq i}\lambda_*(\mathrm{Fil}^jE^{(1)})$. Define the map ψ by the following composition

$$C \otimes I^{i+b} \xrightarrow[\simeq]{\varphi^{-1}_{R^{\leq i}\lambda_*}} \mathrm{Fil}^{i+b}C^{(1)} \xrightarrow{\mathrm{can}} C^{(1)}.$$

The first composition formula then follows from the following enlarged commutative diagram

$$\begin{tikzcd} C \otimes I^{i+b} \arrow[r, "\varphi^{-1}_{R^{\leq i}\lambda_*}", "\simeq"'] \arrow[dr, "\psi"'] & \mathrm{Fil}^{i+b}C^{(1)} \arrow[r, "\varphi", "\simeq"'] \arrow[d, "\mathrm{can}"] & I^{i+b} \otimes C \arrow[d, "\mathrm{can}"] \\ & C^{(1)} \arrow[r, "\varphi"] & C. \end{tikzcd}$$

For the second formula, it follows from the observation that the composition below is the canonical map

$$I^{i+b} \otimes C^{(1)} \xrightarrow{\varphi} I^{i+b} \otimes C \xrightarrow[\simeq]{\varphi^{-1}_{R^{\leq i}\lambda_*}} \mathrm{Fil}^{i+b}C^{(1)}.$$

□

Recall that evaluating at prisms whose reduction is p-completely flat is t-exact, see Proposition 3.11. By taking all the bounded prisms $(A,I)\in Y_{\Delta}$ such that $\mathrm{Spf}(\overline{A})$ is p-completely flat over Y, the above implies the following result on individual relative prismatic cohomology crystal.

Corollary 5.5. *Let $f:X\to Y$ be a smooth proper morphism of smooth p-adic formal schemes over $\mathcal{O}_K$, and assume $(\mathcal{E},\varphi_{\mathcal{E}})\in \mathrm{F\text{-}Crys}^{\mathrm{coh}}(X)$ is I-torsionfree of height $[a,b]$. For $i\in\mathbb{N}$, the $R^if_{\Delta,*}\mathcal{E}$ is a coherent prismatic F-crystal over Y_{Δ}, such that the image of its Frobenius morphism φ within $\mathcal{I}^a_{\Delta}\otimes R^if_{\Delta,*}\mathcal{E}$ satisfies the inclusions:*

$$\mathrm{Im}(\mathcal{I}_{\Delta}^{b+\min\{i,n\}}\otimes R^if_{\Delta,*}\mathcal{E})\subseteq \mathrm{Im}(\varphi)\subseteq \mathrm{Im}(\mathcal{I}_{\Delta}^{a+\max\{0,i-n\}}\otimes R^if_{\Delta,*}\mathcal{E}).$$

In particular, the I-torsionfree quotient of $R^if_{\Delta,*}\mathcal{E}$ has Frobenius height in $[a+\max\{0,i-n\},b+\min\{i,n\}]$.

Here we implicitly use Theorem 4.37.(2) to identify $R^if_{\Delta,*}\mathcal{E}^{(1)}$ with $\varphi^*_{Y_{\Delta}}R^if_{\Delta,*}\mathcal{E}$ for the Frobenius morphism of $R^if_{\Delta,*}\mathcal{E}$.

The following lemma says that passing to I-power torsions or I-torsionfree quotient preserves F-crystals.

Lemma 5.6. *Let X be a smooth p-adic formal schemes over $\mathcal{O}_K$, and let $(\mathcal{E},\varphi_{\mathcal{E}})\in \mathrm{F\text{-}Crys}^{\mathrm{coh}}(X)$ be a coherent F-crystal. Then $\mathcal{F}:=\mathcal{E}[I^\infty]$ is (I,p)-power-torsion, hence $\varphi^*_{X_{\Delta}}\mathcal{F}[1/I]=0=\mathcal{F}[1/I]$. In particular, we have a coherent F-crystal $(\mathcal{F},0)$, and the $\varphi_{\mathcal{E}}$ induces a Frobenius isogeny on $\mathcal{E}/\mathcal{F}$ making it an I-torsionfree coherent F-crystal.*

Proof. On Breuil–Kisin prisms $(A,I)\in X_{\Delta}$, we have $\mathcal{F}(A)=\mathcal{E}(A)[I^\infty]$. It suffices to know that $\mathcal{F}(A)$ is p-power torsion. This follows from [DLMS22, Proposition 4.13]: the authors showed that $\mathcal{E}(A)[p^{-1}]$ is a finite projective $A[p^{-1}]$-module, hence $\mathcal{F}(A)[p^{-1}]=0$. □

Next we show the derived pushforward of I-power torsion prismatic F-crystals will have isogenous Frobenius, for trivial reasons.

Proposition 5.7. *Let $f\colon X\to Y$ be a qcqs smooth morphism of smooth p-adic formal schemes over $\mathcal{O}_K$, and let $(\mathcal{E},\varphi_{\mathcal{E}})\in \mathrm{F\text{-}Crys}^{\mathrm{coh}}(X)$ be an I^∞-torsion F-crystal on X_{Δ}. Then both $Rf_{\Delta,*}\mathcal{E}[1/I]=0$ and $\varphi^*_{Y_{\Delta}}Rf_{\Delta,*}\mathcal{E}[1/I]=0$. In particular, we have $(Rf_{\Delta,*}\mathcal{E},Rf_*(\varphi_{\mathcal{E}}))\in \mathrm{F\text{-}Crys}^{\mathrm{perf}}(Y)$.*

Proof. The claim about $Rf_{\Delta,*}\mathcal{E}[1/I]=0$ follows immediately from the assumption that $E[1/I]=0$ and the map being qcqs. Below we show the other claim. By standard argument, we are immediately reduced to the case of $Y=\mathrm{Spf}(R_0)$ is an affine and $X=\mathrm{Spf}(R)$ is also an affine which moreover admits an étale chart: Namely we may assume that there is an étale map $R_0\langle T_i^{\pm1}\rangle\to R$.

Mimicking [DLMS22, Example 3.4], we see that one can find a Breuil–Kisin prism (A,I) covering Y_{Δ} as well as a Breuil–Kisin prism (B,J) covering the relative prismatic site $(X/A)_{\Delta}$, and moreover the relative Frobenius $\varphi_{B/A}\colon B\widehat{\otimes}_{A,\varphi_A}A\to B$ is faithfully flat. Similar to Proposition 3.7, the object $(B,J=IB)$ is weakly final in $(X/A)_{\Delta}$, and its $(i+1)$-fold self product in $(X/A)_{\Delta}$ exist for all $i\in\mathbb{N}$ and are all given by affine objects denoted as $(B^{(i)},IB^{(i)})$. With the above notation, the vanishing we are after translates[8] to:

$$\lim_{\bullet\in\Delta}\mathcal{E}(B^{(\bullet)},IB^{(\bullet)})\widehat{\otimes}_{A,\varphi_A}A[1/I]=0.$$

To that end, it suffices to show $\mathcal{E}(B^{(\bullet)},IB^{(\bullet)})\widehat{\otimes}_{A,\varphi_A}A[1/I]=0$, and in fact it suffices to show this when $\bullet=0$ (as the latter ones are base changed from this case). Finally, since the relative Frobenius on $B=B^{(0)}$ is faithfully flat, we are reduced to knowing $\mathcal{E}(B,IB)\widehat{\otimes}_{B,\varphi_B}B[1/I]=0$. But this follows from the Frobenius isogeny property of $\mathcal{E}$:

$$\mathcal{E}(B,IB)\widehat{\otimes}_{B,\varphi_B}B[1/I]\xrightarrow[\varphi_{\mathcal{E}}]{\cong}\mathcal{E}(B,IB)[1/I]$$

and the assumption that $\mathcal{E}$ is I^∞-torsion (so the latter module in the above equation is 0).

As for the last sentence, we just note that (see for instance [GR22, Proposition 5.11]) $Rf_{\Delta,*}\mathcal{E}$ is a prismatic crystal in perfect complex over Y. □

[8]To see this translation, we refer readers to the discussion around [BS22, 4.17-4.18 and their footnote 10.]

Now we are ready to generalize the “Frobenius isogeny property” and “weak étale comparison” in [GR22].

Theorem 5.8. *Let $f\colon X \to Y$ be a smooth proper morphism between smooth formal schemes over* $\mathrm{Spf}(\mathcal{O}_K)$*, then derived pushforward of F-crystals in perfect complexes on X are F-crystals in perfect complexes on Y, moreover the following diagram commutes functorially:*

$$\begin{tikzcd} \text{F-Crys}^{\text{perf}}(X) \arrow[r, "T(-)"] \arrow[d, "Rf_{\Delta,*}"'] & D^{(b)}_{lisse}(X_\eta, \mathbb{Z}_p) \arrow[d, "Rf_{\eta,*}"] \\ \text{F-Crys}^{\text{perf}}(Y) \arrow[r, "T(-)"] & D^{(b)}_{lisse}(Y_\eta, \mathbb{Z}_p). \end{tikzcd}$$

Proof. Let $(\mathcal{E}, \varphi_{\mathcal{E}})$ be in $\text{F-Crys}^{\text{perf}}(X)$. It is known (see for instance [GR22, Proposition 5.11]) that $Rf_{\Delta,*}\mathcal{E}$ is a prismatic crystal in perfect complexes over Y. To see the induced Frobenius on $Rf_{\Delta,*}\mathcal{E}$ must be an isogeny, we first reduce ourselves to the case where $(\mathcal{E}, \varphi_{\mathcal{E}}) \in \text{F-Crys}^{\text{coh}}(X)$. By considering the I^∞-torsions of $\mathcal{E}$ and its I-torsionfree quotient, which are coherent F-crystals by Lemma 5.6, we are done thanks to Proposition 5.7 and Corollary 5.4 respectively.

To see the commutative diagram, we recall by construction in [BS23, Cor. 3.7] that the étale realization $T(Rf_{\Delta,*}\mathcal{E})$ is isomorphic to the sheaf of complexes over $Y_{\eta,\text{proét}}$, sending an affinoid perfectoid Huber pair $(S[1/p], S)$ over Y_η onto the following complexes

$$\mathrm{fib}\left((Rf_{\Delta,*}\mathcal{E}[1/I]^\wedge_p)(\mathrm{A}_{\inf}(S), I) \xrightarrow{\varphi-\mathrm{id}} (Rf_{\Delta,*}\mathcal{E}[1/I]^\wedge_p)(\mathrm{A}_{\inf}(S), I), \right)$$

namely

$$T(Rf_{\Delta,*}\mathcal{E}) : (S[1/p], S) \longmapsto \Big((Rf_{\Delta,*}\mathcal{E}[1/I]^\wedge_p)(\mathrm{A}_{\inf}(S), I)\Big)^{\varphi=1}.$$

On the other hand, by applying derived global section at weak étale comparison in [GR22, Thm. 6.1], for any perfect prism (A, I) in Y_Δ, there is a natural isomorphism of $\mathbb{Z}_p$-complexes

$$R\Gamma((X_{\overline{A}})_{\eta,\text{proét}}, T(\mathcal{E})) \simeq \Big((Rf_{\Delta,*}\mathcal{E}[1/I]^\wedge_p)(A, I)\Big)^{\varphi=1},$$

where $X_{\overline{A}}$ is the complete base change $X \times_Y \mathrm{Spf}(\overline{A})$. As a consequence, by taking the inverse system with respect to perfect prisms associated to affinoid perfectoid Huber pairs $(S[1/p], S)$ over Y_η, we get a natural isomorphism of $\mathbb{Z}_p$-complete complexes over Y_η

$$Rf_{\eta,*}T(\mathcal{E}) \simeq T(Rf_{\Delta,*}\mathcal{E}).$$

□

Combining the above with Lemma 3.16, we get the following slight refinement of the “weak étale comparison” [GR22, Theorem 6.1].

Corollary 5.9. *Let $f\colon X \to Y$ be a smooth proper morphism between smooth formal schemes over* $\mathrm{Spf}(\mathcal{O}_K)$*, and let $(\mathcal{E}, \varphi_{\mathcal{E}}) \in \text{F-Crys}^{\text{perf}}(X)$. Then $T(R^i f_{\Delta,*}\mathcal{E}) = R^i f_{\eta,*}(T(\mathcal{E}))$.*

References

[ALB23] Anschütz, Johannes and Arthur-César Le Bras, *Prismatic Dieudonné theory*, Forum of Mathematics, Pi **11** (2023), e2.

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171. MR 751966

[Bha22] Bhargav Bhatt, *Prismatic F-gauges*, 2022, Lecture notes available at `https://www.math.ias.edu/~bhatt/teaching/mat549f22/lectures.pdf`.

[BL22a] Bhargav Bhatt and Jacob Lurie, *Absolute prismatic cohomology*, arXiv e-prints (2022), arXiv:2201.06120.

[BL22b] ———, *The prismatization of p-adic formal schemes*, arXiv e-prints (2022), arXiv:2201.06124.

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze, *Integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci. **128** (2018), 219–397. MR 3905467

[BMS19] ———, *Topological Hochschild homology and integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci. **129** (2019), 199–310. MR 3949030

[BS17] Bhargav Bhatt and Peter Scholze, *Projectivity of the Witt vector affine Grassmannian*, Invent. Math. **209** (2017), no. 2, 329–423. MR 3674218

[BS22] ———, *Prisms and prismatic cohomology*, Ann. of Math. (2) **196** (2022), no. 3, 1135–1275. MR 4502597

[BS23] ———, *Prismatic F-crystals and crystalline Galois representations*, Camb. J. Math. **11** (2023), no. 2, 507–562. MR 4600546

[DL21] Heng Du and Tong Liu, *A prismatic approach to $(\varphi, \hat{G})$-modules and F-crystals*, arXiv e-prints (2021), arXiv:2107.12240.

[DLMS22] Heng Du, Tong Liu, Yong Suk Moon, and Koji Shimizu, *Completed prismatic F-crystals and crystalline $\mathbf{Z}_p$-local systems*, arXiv e-prints (2022), arXiv:2203.03444.

[Dri20] Vladimir Drinfeld, *Prismatization*, arXiv e-prints (2020), arXiv:2005.04746.

[GL21] Haoyang Guo and Shizhang Li, *Period sheaves via derived de Rham cohomology*, Compos. Math. **157** (2021), no. 11, 2377–2406. MR 4323988

[GR22] Haoyang Guo and Emanuel Reinecke, *A prismatic approach to crystalline local systems*, arXiv e-prints (2022), arXiv:2203.09490.

[Ked06] Kiran S. Kedlaya, *Fourier transforms and p-adic 'Weil II'*, Compos. Math. **142** (2006), no. 6, 1426–1450. MR 2278753

[LL20] Shizhang Li and Tong Liu, *Comparison of prismatic cohomology and derived de Rham cohomology*, arXiv e-prints (2020), arXiv:2012.14064, to appear in J. Eur. Math. Soc.

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659

[Lur17] Jacob Lurie, *Higher algebra*, 2017, available at `https://www.math.ias.edu/~lurie/papers/HA.pdf`.

[Lur18] ———, *Spectral algebraic geometry*, 2018, available at `https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf`.

[Mag24] Kirill Magidson, *Divided powers and derived de rham cohomology*, 2024.

[Mat16] Akhil Mathew, *The Galois group of a stable homotopy theory*, Adv. Math. **291** (2016), 403–541. MR 3459022

[Mat22] ———, *Faithfully flat descent of almost perfect complexes in rigid geometry*, J. Pure Appl. Algebra **226** (2022), no. 5, Paper No. 106938, 31. MR 4332074

[MT20] Matthew Morrow and Takeshi Tsuji, *Generalised representations as q-connections in integral p-adic Hodge theory*, arXiv e-prints (2020), arXiv:2010.04059.

[MW21] Yu Min and Yupeng Wang, *Relative (φ, Γ)-modules and prismatic F-crystals*, arXiv e-prints (2021), arXiv:2110.06076.

[MW22] ———, *P-adic Simpson correpondence via prismatic crystals*, arXiv e-prints (2022), arXiv:2201.08030.

[Shi18] Koji Shimizu, *Constancy of generalized Hodge-Tate weights of a local system*, Compos. Math. **154** (2018), no. 12, 2606–2642. MR 3873527

[Sta23] The Stacks Project Authors, *Stacks Project*, `https://stacks.math.columbia.edu`, 2023.

[Tan22] Longke Tang, *Syntomic cycle classes and prismatic Poincaré duality*, arXiv e-prints (2022), arXiv:2210.14279.

[Tia21] Yichao Tian, *Finiteness and Duality for the cohomology of prismatic crystals*, arXiv e-prints (2021), arXiv:2109.00801.

[Wu21] Zhiyou Wu, *Galois representations, (φ, Γ)-modules and prismatic F-crystals*, Doc. Math. **26** (2021), 1771–1798. MR 4353339

H. Guo: The University of Minnesota Twin cities, Vincent Hall, 206 Church St. SE, Minneapolis, MN 55455, USA

Email address: `ghy@umn.edu`

S. Li: Morningside Center of Mathematics and State Key Laboratory of Mathematics Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

Email address: `lishizhang@amss.ac.cn`