Notes on GIT

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GIT

Goal: ① Moduli problem in AG.

② Equivariant geometry.

1) G = reductive gp. $/\mathbb{C}$

2) linear action of G on $\mathbb{P}^n$

3) $X \hookrightarrow \mathbb{P}^n$ q-proj, equivariant for action of G.

Guiding principle: "equivariant" constructions shouldn't depend on quotient description X/G.

③ moduli of vector bundles on a curve

pathology: · highly non-separated.

· too many vector bundles.

④ results: restrict to $\mathcal{M}_{2,d}$.

1) Atiyah-Bott formula, $P_t(-) = \sum_{i\geq 0} t^i \dim H^i(-;\mathbb{Q})$.

$$P_t(\mathcal{M}^{ss}_{2,d}) = P_t(\mathcal{M}_{2,d}) - \sum_{k>\frac{d}{2}} t^{\#_k} P_t(\mathcal{M}_{1,k}) P_t(\mathcal{M}_{1,d-k})$$

$$= \frac{(1+t)^{2g}(1+t^3)^{2g}}{(1-t^2)^2(1-t^4)} - \sum_{k>\frac{d}{2}} t^{\#_k} \left(\frac{(1+t)^{2g}}{1-t^2}\right)^2$$

$\#_k = 2k - d + g + 1$.

2) $\exists$ unique positive generator of $\mathrm{Pic}(\mathcal{M}^{ss}_{2,0})$: $\mathcal{L}$.

Verlinde formula: $H^i(\mathcal{M}^{ss}_{2,d}, \mathcal{L}^{\otimes k}) = 0$ for $i>0$.

$$\dim H^0(\mathcal{M}^{ss}_{2,d}, \mathcal{L}^{\otimes k}) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \left(\sin\left(\frac{j\pi}{k+2}\right)\right)^{2-2g}.$$

/field k.

Defn Linear alg. gp. is a smooth affine gp scheme $/k$.

ex. $GL_n = \mathrm{Spec}\, k[a_{ij}][\frac{1}{\det}]$

$\mathbb{G}_m = \mathrm{Spec}\, k[t^{\pm 1}]$, $t \longmapsto t_1 \otimes t_2$

Split torus $T \cong (\mathbb{G}_m)^n$.

Defn a torus is a linear alg. gp T, s.t. $T_{\bar{k}} \cong (\mathbb{G}_m)^n_{\bar{k}}$.

Weil restriction. given $\pi: X \to Y$ finite flat, given W/X, one can construct $\pi_* W/Y$, s.t. $\pi_* W(Y') = W(Y' \times_Y X)$.

e.g. W is total space of a coherent sheaf $\mathcal{F}$

$\pi_* W$ is total space of $\pi_* \mathcal{F}$.

Deligne Torus: $\pi_*((\mathbb{G}_m)_{\mathbb{C}}) = \mathbb{S}$. $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$

$k[\mathbb{S}] = (\mathbb{C}[z^{\pm 1}, \bar{z}^{\pm 1}])^{\sigma=1}$ $\sigma: \lambda \mapsto \bar{\lambda}$, $z \mapsto \bar{z}$

$= \mathbb{R}[a, b, \frac{1}{a^2+b^2}]$. $(z \longmapsto a+bi,\ \bar{z} \longmapsto a-bi)$.

e.g. A is a geom'lly reduced finite k-alg.

T = Weil restriction of $(\mathbb{G}_m)_A$ along $\mathrm{Spec}\,A \to \mathrm{Spec}\,k$

$T(k) = A^\times$. $T \hookrightarrow GL(A) = GL_{\dim(A)}$

gives example of "max'l torus" in GL_n which non-split.

Repn of G

1) $G \longrightarrow GL(V) = GL_n$

2) $G \times V \longrightarrow V$ (linear ...)

3) most useful: comodules over $k[G]$.

Defn (comodule) $\rho: V^* \longrightarrow V^* \otimes k[G]$

axioms: 1) $V^* \longrightarrow V^* \otimes k[G] \xrightarrow{1 \otimes \epsilon} V^*$, id.

2)
$$\begin{array}{ccc} V^* & \xrightarrow{\rho} & V^* \otimes k[G] \\ \downarrow \rho & & \downarrow 1\otimes\Delta \\ V^* \otimes k[G] & \xrightarrow{\rho \otimes 1} & V^* \otimes k[G] \otimes k[G] \end{array}$$
(associativity)

Ex. 1) Cat. of $\mathbb{G}_m$-repn's is eq. to (opposite) cat. of $\mathbb{Z}$-graded vector spaces. $\rho: V \longrightarrow V \otimes k[t^{\pm 1}]$.

$v \longmapsto \sum v_i t^i$

by associativity, $\rho(v_i) = v_i t^i$.

Hence if we define $V_i = \mathrm{Span}\{\rho(v)_i \text{ for } v \in V\}$.

we have $V \cong \bigoplus V_i$.

2) split torus T, consider $\mathrm{Hom}_{gp}(T, \mathbb{G}_m) = M$, "character lattice".

any rep V of $T \cong \bigoplus_{\chi \in M} V_\chi$, where T acts as $\chi(t)$ on V_χ.

3) Deligne Torus: $\mathbb{S} \longleftrightarrow \mathbb{G}_{m,\mathbb{R}}$

$\mathbb{R}[a, b, \frac{1}{a^2+b^2}] \longrightarrow \mathbb{R}[t, t^{-1}]$.

$a \mapsto$ ~~$a+b$~~ t

$b \longmapsto 0$.

$\Rightarrow$ reps of $\mathbb{S}$ will have an $\mathbb{R}$-v.s. structure

w/ grading $V = \bigoplus_w V_w$.

along with splitting $V_{\mathbb{C}} \cong \bigoplus (V_{\mathbb{C}})^{a,b}$ s.t.

$(V_w)_{\mathbb{C}} = \bigoplus_{a+b=w} (V_{\mathbb{C}})^{a,b}$ and

$(V_{\mathbb{C}})^{a,b} = \overline{(V_{\mathbb{C}})^{b,a}}$

$\Rightarrow$ reps of $\mathbb{S}$ are eq. to $\mathbb{R}$-Hodge structure.

Prop any repn V of G is a union of fin. dim'l sub co-modules.

pf. $v \in V$ lies in finite dim'l co-module

choose $\{e_i\}$ for $k[G]$

$\rho(v) = \sum_{\text{finite}} v_i \otimes e_i$ Claim: linear span of v_i is a sub co-module.

Indeed, $\Delta(e_i) = \sum r_i^{jk} e_j \otimes e_k$

$\sum \rho(v_i) \otimes e_i = \sum r_i^{jk} v_i \otimes e_j \otimes e_k$

so $\rho(v_k) = \sum_{i,j} r_i^{jk} v_i \otimes e_j \in \mathrm{Span}(v_i) \otimes k[G]$.

Main Structure thms.

1) $G \hookrightarrow GL_n$ for some n. (find a finite sub-repn of $V = k[G]$ containing a set of generators).

2). Jordan decomp.: for $g \in G(\bar{k})$

$\exists!$ $g = g_{ss} \cdot g_u$. s.t. g_{ss}, g_u commute.

so we can define unipotent and solvable subgps.

3) connected solvable subgp B is max'l iff G/B is proj.
called Borels, and they always exist.
over $\bar{k}$, Borels are unique up to conjugate.

Defn. P is called parabolic subgp if G/P is proj.

Fact any parabolic subgps contains ~~some~~ one Borel subgp.

4) $\exists$ max'l torus $T \hookrightarrow G$ s.t. $T_{\bar{k}} \hookrightarrow G_{\bar{k}}$ is max'l.
it's unique up to conjugate.

5) $\exists!$ unique max'l normal unipotent $R_u(G) \hookrightarrow G \twoheadrightarrow H$
called unipotent radical.

Defn G is reductive if $R_u(G) = 1$

6) if char. $k = 0$, Reductive $\Leftrightarrow$ linearly reductive

Defn linearly reductive means the following equivalent things hold:
1) cat. of $\mathrm{Rep}(G)$ has no higher ext's
2) $V \twoheadrightarrow W$ surjection of G-reps always has a splitting.

Nagata: in char. p, linearly reductive $\Leftrightarrow$ $G_0 = (\mathbb{G}_m)^n$ and
$p \nmid |G/G_0|$.

Lemma restriction : $Sh((Sch/k)_\tau) \longrightarrow Sh((Ring^{op}/k)_\tau)$
is an equivalence for $\tau = \{Zar., \acute{e}t., fppf\}$

F: $Rings/S \to Sets$, a sheaf is called locally finitely presented
if $\forall$ filtered system R_i, $\varinjlim F(R_i) \xrightarrow{\sim} F(\varinjlim R_i)$.

Fact a scheme X/S is l.f.p. iff h_X is l.f.p.

Fact. $Sh(Rings^{op}/k) \underset{L}{\overset{res}{\rightleftarrows}} Sh(Rings^{op}_{f.t.}/k)$
restriction has a left adjoint L which is fully faithful,
$F \xleftarrow{\simeq} L(res(F))$ iff F is l.f.p.
"LFP sheaves are functorially determined by values on f.t. rings".

$G \times X \xrightarrow{\sigma} X$ is a gp action, consider
$\chi: G \times X \xrightarrow{\sigma \times p_2} X \times X$.
Given S-pt $S \xrightarrow{f} X$, get $\chi_f: G \times S \xrightarrow{\sigma \times p_2 \circ (id, f)} X \times S$

e.g. $\mathrm{Spec}(k) \xrightarrow{f} X$, χ_f is a orbit map $G \longrightarrow X$.
In general

$$\begin{tikzcd} \mathrm{Stab}_f \arrow[r] \arrow[d] & G \times S \arrow[d, "\chi_f"] \\ S \arrow[r, "f \times id"] & X \times S \end{tikzcd}$$

$\mathrm{Stab}_f \subseteq G \times S$ is a subgp scheme

geometric fibers ~~are~~ is stablizers of corresponding point of X.

• Fiber dim'n of $\mathrm{Stab}_g \longrightarrow S$ is upper semi-cont.

pf. general fact: $X \xrightarrow{f} Y$, then $\dim f^{-1}(f(x))$ is upper-semicont.
Now we restrict this upper semi-cont. along section.

e.g. $\mathbb{G}_m \curvearrowright \mathbb{A}^1$ $\quad \mathrm{Stab}_{\mathrm{id}_{\mathbb{A}^1}} = k[z, z^{-1}, x]/(zx - x) \longleftarrow k[x]$.

• If X Noetherian w/ finite Krull dim'n:
$$\dim(G) = \dim(G\cdot x) + \dim(G_x)$$
for any point $x \in X$

• closure $\overline{G\cdot x}$ is a union of lower dim'l orbit, if an orbit in there has minimal dim'n, then it's closed.

$G \curvearrowright \operatorname{Spec} R \rightsquigarrow$ giving R a G-mod. structure cptible w/ ring structure.

e.g. T is split torus $\curvearrowright R$ $\Leftrightarrow$ M-grading on $R = \bigoplus_{\chi \in M} R_\chi$
(M: char. lattice)

Lemma. if $G \curvearrowright \mathrm{Spec}(R)$ and R is finite type, then $\exists$ eq. embd $\mathrm{Spec}(R) \hookrightarrow V$ for some linear rep. V.

pf. one just find some finite dim'l sub G-repn V^* in R containing all the generators, then $k[V^*] \twoheadrightarrow R$.

Rmk. Matsushima's thm: G reductive, $H \subseteq G$ is a closed subg, then G/H is affine $\Leftrightarrow$ H is reductive.

Consequence: $G \hookrightarrow GL_n$, if G is reductive, then GL_n/G is affine, hence $GL_n/G \hookrightarrow V \Rightarrow G = \mathrm{Stab}_{GL_n}(v)$.
$1 \xmapsto{GL_n\text{-eq}} v$

$G \curvearrowright V \rightsquigarrow G \curvearrowright \mathbb{P}(V)$.
consider G-equivariant $X \hookrightarrow \mathbb{P}(V)$.

$$\left(\begin{tikzcd} G\times X \arrow[r] \arrow[dr, dashed] & \mathbb{P}(V) \\ & X \arrow[u, hook] \end{tikzcd}\right)$$
it's called G-q-proj. schemes.

For torus action, there's following thm:

Thm (Sumihiro) T torus, X is T-q-proj, then X is covered by T-equivariant open affines.

pf. • if $k = k^{sep}$, $X \hookrightarrow \mathbb{P}(V)$ is closed, then it suffices to do for $\mathbb{P}(V)$, and we cover it by $\mathbb{P}(V)_f$, where $f \in V^*$ is an eigenvector for T-action.
• if $k = k^{sep}$, reduce to $U \subseteq \mathrm{Spec}(A)$, then $I_{\mathrm{Spec}(A)\setminus U}$ is T-equiv.
• in general, any affine of $X_{k^{sep}}$ is defined over $X_{k'}$, for some k'/k finite Galois. And one could take $\bigcap_{\sigma \in \mathrm{Gal}(k'/k)} U^\sigma$.

Thm (fixed pts) Let X be a T-q-proj scheme, then $X^T \hookrightarrow X$ is a closed subscheme, smooth if X is smooth.

$(X^T(\mathrm{Spec}\,R) = \mathrm{Map}_T(\mathrm{Spec}(R), X))$. R w/ trivial T-action.

pf: enough to check for affine $X = \mathrm{Spec}(A)$, $k = \bar{k}$.

define $B = A/A\cdot(\bigoplus_{\chi\neq 0} A_\chi)$, $Z = \mathrm{Spec}(B)$ represents X^T.

smoothness: $T_x(\mathrm{Spec}(B)) = (T_x X)^T$ for $x \in \mathrm{Spec}\,B$.

if X is smooth at x, then one can lift every non-zero eigenvector in $m_{X,x}/m^2_{X,x}$ to an eigenvector in $m_{X,x}$, liftings of basis would cut out Z in a nbhd of x.

Thm (Białynicki-Birula) $X \hookrightarrow \mathbb{P}(V)$ is $\mathbb{G}_m$-q-proj., then

(1) $Y(R) := \mathrm{Map}_{\mathbb{G}_m}(\mathbb{A}^1 \times \mathrm{Spec}(R), X)$ is a scheme.

(2) (a) restriction: $Y(R) \xhookrightarrow{i} \mathrm{Map}(\{1\}\times \mathrm{Spec}(R), X)$ is a local immersion.

(b) $X^T(R) \xrightarrow{\sigma} Y(R)$ is a closed embedding

(c) $\pi: Y(R) \longrightarrow X^T(R)$ by $|_{\{0\}\times \mathrm{Spec}(R)}$ is affine

(3) if X is smooth, then so is Y, and $\pi: Y \to X$ is an étale locally trivial bundle of affine space ($\mathbb{A}^n$).

e.g. $\mathbb{G}_m \curvearrowright V$ repn. $a_0 \le \cdots \le a_n$ $t\cdot[z_0,\dots,z_n] = [t^{a_0}z_0,\dots,t^{a_n}z_n]$.

$$\mathbb{P}(V)^{\mathbb{G}_m} = \bigsqcup_{a_\alpha} \{[0:0:\cdots: z_i,\dots,z_{i+r_i},0\cdots]\}.$$

$$Y = \bigsqcup_{a_\alpha} \{[0:0:\cdots, z_i,\dots,z_{i+r_i},*,\dots,*]\}.$$

Lemma: ~~Rmk~~ (1) $X \hookrightarrow X'$ is $\mathbb{G}_m$-eq. closed immersion, then

$$\begin{tikzcd} Y \arrow[r, hook, "j"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, hook, "j'"] & X' \end{tikzcd}$$

is Cartesian.

(2) $X \hookrightarrow X'$ is $\mathbb{G}_m$-eq. open immersion, then

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[d] & X^{\mathbb{G}_m} \arrow[d] \\ Y' \arrow[r, "\pi'"] & (X')^{\mathbb{G}_m} \end{tikzcd}$$

is Cartesian.

One use (2) to construct $Y^\circ \xrightarrow{\pi^\circ} X^{\mathbb{G}_m}$, and define j locally and see the defn glued to give a global j.

Ex. $\mathbb{A}^2 = \mathbb{A}^1(1) \times \mathbb{A}^1(0)$. BB stratum is all of $\mathbb{A}^2$.

if we remove the origin, $\mathbb{A}^2 - \{0\}$. Then $Y = \mathbb{A}^1 \times \mathbb{G}_m$.

Ex. G itself has an action via conjugation for any $\lambda: \mathbb{G}_m \to G$.

define $P_\lambda := \{g \in G, \text{ s.t. } \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists}\}$

$=$ BB subscheme corresponding to this $\mathbb{G}_m$-action.

Facts · P_λ will be a parabolic subgp (G/P_λ is proper):

$$\begin{tikzcd} \lambda: \mathbb{G}_m \arrow[r] & G \arrow[r, hook] & GL(V) \\ & P_\lambda \arrow[u, "j"] \arrow[r, hook] & PGL_{V,\lambda} \arrow[u] \end{tikzcd}$$

choose eigenbasis for $V = \bigoplus V_\alpha$ $\alpha_0 > \alpha_1 > \cdots > \alpha_k$

$PGL_{V,\lambda} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ So $G/P_\lambda \hookrightarrow \dfrac{GL(V)}{PGL_{V,\lambda}} = \mathrm{Flag}$.

· any parabolic in any reductive gp arises in this way.

Quotients: $H \leq G$ closed subgp
The quotients G/H exist and are quasi-proj. schemes

Thm (Chevalley) if $H \subseteq G$ (lin. gp) is a closed subgp, then $\exists$ repn V and a line $L \subseteq V$, s.t. $H = \mathrm{Stab}(L) \leq PGL(V)$.
Then the G-orbit of $[L]$ in ~~$\mathbb{P}(V)$~~ $PGL(V)$ is G/H.

pf. Step 1. find f.d. H-repn $V \subseteq I_H$ which generates as an ideal, so $H = \mathrm{Stab}(I_H) = \mathrm{Stab}(V)$.
Step 2. find f.d. G-repn $V' \supseteq V$ s.t. G acts faithfully (i.e., $G \hookrightarrow GL(V')$)
we still have $H = \mathrm{Stab}_G(V \leq V')$.
Step 3. use Plücker embedding $G \subset Gr(\dim(V), V') \hookrightarrow \mathbb{P}(\wedge^{\dim(V)} V')$

In general:
Thm $G \subset X$, G lin. alg. gp, X is a scheme, and $G \times X \longrightarrow X \times X$ is a monomorphism.
then X/G is an alg. space.

Rmk. $G \times X \longrightarrow X \times X$ is a monomorphism iff $G(R) \subset X(R)$ freely.

Defn. an alg. space is a sheaf F on $(Sch/S)_{ét}$, s.t.
1) $F \xrightarrow{\Delta} F \times F$ is representable.
2) $\exists$ a surj. étale map: $U \longrightarrow F$ where U is a scheme.

Rmk ① $\Leftrightarrow$ every map $X \times F$ is representable b/c:
$$X \times_F Y = F \times_{(\Delta, F \times F)} (X \times Y)$$

② for flat + f.p. map $f: X \to Y$.
f is surj. $\Leftrightarrow$ ~~$\exists$ $X(R) \to Y(R)$ $\forall$ scheme R.~~ Étale locally f has a section.

Defn. ③ equiv. relation on X is a scheme $R \longrightarrow X \times X$ s.t. $\forall$ A scheme
$R(A) \hookrightarrow X(A) \times X(A)$ is equiv. relation.
a eq. rel. is étale if $R \to X$ is étale.
(e.g. G finite gp $\subset X$ freely, then $G \times X \to X \times X$ is an étale eq. rel.).
for any eq. rel. in schemes $R \to U \times U$, one can form sheaf U/R via sheafification.

Thm. (Tag 04S5) TFAE, a sheaf F on $(Sch/S)_{ét}$,
① F is an alg. space
② $\exists$ rep'able ét surj. $U \to F$ w/ U a scheme
③ $\exists$ an étale eq. rel. $R \to U \times U$ s.t. $F \simeq U/R$.

even harder: these are eq. to ①', ②', ③' where "étale" is replaced by "fppf".

So if $G \circlearrowright X$ freely, then $G \times X \longrightarrow X$ is fppf, hence X/G is an alg. space.

Ex. char $\neq 2$.

$$\begin{array}{ccc} R = \Delta \cup \Delta^{-} & \longrightarrow & \mathbb{A}^1 \times \mathbb{A}^1 \\ \uparrow & \nearrow & \\ R' = \Delta \sqcup (\Delta^{-} \setminus \{0,0\}) & & \end{array}$$

$\mathbb{A}^1/R'$ is <u>not</u> a scheme:

$$\mathbb{A}^1 \longrightarrow \mathbb{A}^1/R' \longrightarrow \mathbb{A}^1/R = \mathbb{A}^1, \qquad \text{composite } x \longmapsto x^2$$

$\mathbb{A}^1/R' \cong$ —:—$/(\mathbb{Z}/2)$, where $\sigma(x) = \begin{cases} -x & \text{if } x \notin \{P, Q\} \\ Q & \text{if } x = P \\ P & \text{if } x = Q \end{cases}$

which as a locally ringed space is not a schem.

Another way to argue is to realize $R' \to \mathbb{A}^1$ is étale, and there's no étale deg. 2 map $f: \mathbb{A}^1 \to X$.

if $f: X \longrightarrow Y$ is a map of alg. spaces, $\Delta_f: X \to X \times_Y X$ will always be representable, locally f.p., monomorphism, separated, locally quasi-finite.

Defn. f is separated if Δ_f is closed, qs if it's qc.

Terrible Ex. $\mathbb{Z} \circlearrowright \mathbb{A}^1 / \mathbb{Q}$, one can form $\mathbb{A}^1/\mathbb{Z} \leftarrow \mathbb{A}^1 \leftleftarrows \mathbb{Z} \times \mathbb{A}^1$

It's not qs $\rightsquigarrow$ provides counterexample of the following:

Thm. X is a qcqs alg. space $/k$, then $\exists$ dense open subscheme $X' \subseteq X$.

Cor. if a linear gp G acts freely on a qc scheme X, then $\exists$ dense open G-invariant subscheme U, s.t. U/G is a scheme.

pf. in this case, diagonal map is always qc., b/c G is qc.

$$\begin{array}{ccccc} G \times X & \rightrightarrows & X & \longrightarrow & Y = X/G \\ & & \uparrow & \lrcorner & \uparrow \\ & & U & \longrightarrow & Y^{\circ} \end{array}$$

Defn X is a scheme, a principal G-bundle on X is an alg. space $Y \circlearrowleft G$, s.t. (1) $G \times Y \xrightarrow{\simeq} Y \times_X Y$

$$\begin{array}{c} Y \\ \downarrow \pi \\ X \end{array}$$

(2) étale locally, π admits a section.

(it's equivalent to a sheaf w/ properties (1) & (2).

Lemma. If G is affine, any principal G-bundle is a scheme, affine $/X$

pf.
$$\begin{tikzcd} G\times U \arrow[r] \arrow[d] & Y \arrow[d] \\ U \arrow[r, "\text{ét}"] & X \end{tikzcd}$$
, as $G\times U \to U$ is affine we have $Y \to X$ is affine by étale (fppf) descent.

Rmk for certain gps, "special gps", Principal G-bundles are always Zariski locally trivial.

Ex. for GL_n, $\exists$ equivalence of cats (groupoids)

$$\{GL_n\text{-bundles}/X\} \cong \{\text{vector bundles}/X\}$$

$$(\text{Trivializations of } \mathcal{E}/U \to U) = Fr(\mathcal{E}) \longleftarrow\!\shortmid\ \mathcal{E}$$

$$Y \longmapsto \mathbb{A}^n \times_{GL_n} Y.$$

Ex. $B \subseteq G$ Borel, is special.

Return to quotients $G \underset{\text{free}}{\circlearrowright} X \rightsquigarrow$ quotient space X/G.

$$G \circlearrowright U \leftsquigarrow (X/G)^\circ \subseteq X/G$$

s.t. U/G is a scheme.

formally $U \to U/G$ is a principal G-bundle

$\underset{(G \text{ affine})}{\Longrightarrow}$ this map is affine

Conclusion: If X is a scheme w/ free G-action ~~for~~ then

$\exists$ dense open subscheme of X covered by G-equiv. open affines $U_\alpha = \mathrm{Spec}(R_\alpha)$ s.t. $U_\alpha/G_\alpha = \mathrm{Spec}(R_\alpha^G)$ is affine.

$\therefore$ X/G is a scheme iff X admits open affine cover of this form.

Stacks: what if $G \circlearrowright X$ is not free?

$G\times X \to X\times X$ is not a monomorphism (it's a gpoid!)

Defn. Groupoid scheme is a diagram

$$\begin{tikzcd} X_1 \times_{s, X_0, t} X_1 \arrow[r] & X_1 \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & X_0 \arrow[l, bend left, "e"] \end{tikzcd}$$

is a groupoid.

$(\gamma_1, \gamma_2) \longmapsto \gamma_1\gamma_2$ s.t. apply to any Y, this diagram

Defn. A <u>Morita</u> morphism is a map of gpoids

$$f_\cdot : X_\cdot \to Y_\cdot$$

s.t. ① $f_0 : X_0 \to Y_0$ is fppf.

②
$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d] \\ X_0\times X_0 \arrow[r] & Y_0 \times Y_0 \end{tikzcd}$$

Observation if $f_\cdot : X_\cdot \to Y_\cdot$ is a Morita morphism w/ a section of $f_0 : X_0 \to Y_0$ (σ), then $\exists$ functor: $\sigma_\cdot : Y_\cdot \to X_\cdot$

s.t. $f_\cdot \circ \sigma_\cdot = id$

$\exists\ \theta :$ $\sigma_\cdot \circ f_\cdot \Rightarrow id_X$

ex. • $X \to Y$ fppf. Let $X_1 = X_0 \times_Y X_0$.

• $G \curvearrowright X$ and $H \supseteq G$ subgp. "$X/G \cong H \times_G X / H$".

$$\begin{tikzcd}
G\times X \arrow[d, shift left] \arrow[d, shift right] & H\times G\times X \arrow[l] \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & H\times(H\times_G X) \arrow[d, shift left] \arrow[d, shift right] \\
X & H\times X \arrow[l] \arrow[r] & H\times_G X
\end{tikzcd}$$

Morita Morita

"X/G" "$H\times X/H\times G$" "$H\times_G X/H$"

Given gpoid scheme $X_1 \underset{s}{\overset{t}{\rightrightarrows}} X_0 =: X_\bullet$.

define a cat. $\mathrm{Qcoh}(X_\bullet)$ as follows

obj: (1) $\mathcal{E}$ a q-coh. sheaf on X_0

(2) an isom. $\alpha: t^*\mathcal{E} \to s^*\mathcal{E}$

s.t. $X_0 \underset{s}{\overset{t}{\leftleftarrows}} X_1 \underset{p_2}{\overset{p_1}{\Lleftarrow}} X_2 = X_1 \times_{X_0} X_1 =$ "$\{\bullet \to \bullet \to \bullet\}$".

(A). $p_1^* t^*\mathcal{E} = c^*t^*\mathcal{E} \xrightarrow[\simeq]{c^*(\alpha)} c^*s^*(\mathcal{E}) = p_2^* s^*(\mathcal{E})$

$$\begin{tikzcd}
p_1^*t^*\mathcal{E} \arrow[rr, "c^*(\alpha)"] \arrow[dr, "p_1^*(\alpha)"'] & & p_2^*s^*(\mathcal{E}) \\
& p_1^*s^*(\mathcal{E}) = p_2^*t^*(\mathcal{E}) \arrow[ur, "p_2^*(\alpha)"'] &
\end{tikzcd}$$

(B) $e^*\alpha: \mathcal{E} \to \mathcal{E}$ is identity.

Rmk for $G\times G\times X \Rrightarrow G\times X \rightrightarrows X$

$p_1(g_1,g_2,x) = (g_2,x)$ $t(g,x) = x$

$c(g_1,g_2,x) = (g_1g_2,x)$ $s(g,x) = g\cdot x$.

$p_2(g_1,g_2,x) = (g_1, g_2x)$

$\forall (g,x)$. $\alpha: \mathcal{E}_x \xrightarrow{\simeq} \mathcal{E}_{gx}$.

$\forall (g_1,g_2,x)$ $\mathcal{E}_x \xrightarrow{\simeq} \mathcal{E}_{g_2x} \xrightarrow{\simeq} \mathcal{E}_{g_1(g_2x)}$, $\mathcal{E}_x \xrightarrow{\simeq} \mathcal{E}_{(g_1g_2)\cdot x}$

Example $\mathrm{Qcoh}(G \rightrightarrows \cdot) \cong \mathrm{Rep}(G)$.

$\mathrm{Hom}(\mathcal{E},\mathcal{F}) = \{f: \mathcal{E} \to \mathcal{F}$ on X_0, s.t.
$$\begin{tikzcd}
t^*\mathcal{E} \arrow[r, "\alpha"] \arrow[d, "t^*f"'] \arrow[dr, phantom, "\circlearrowright"] & s^*\mathcal{E} \arrow[d, "s^*f"] \\
t^*\mathcal{F} \arrow[r, "\beta"] & s^*\mathcal{F}
\end{tikzcd}$$
$\}$

More Examples $\mathbb{P}(V)$, consider $\mathcal{O}(1)$, is it canonically equivariant w.r.t. $PGL(V)$?

Ans: no. b/c $PGL(V)\times\mathbb{P}(V) \rightrightarrows \mathbb{P}(V)$.

fixing pt in $\mathbb{P}(V)$, get orbit map

$$\begin{tikzcd}
PGL(V) \arrow[r, "s"] \arrow[d, "t"'] & \mathbb{P}(V) \\
\{x\} \arrow[ur] &
\end{tikzcd}$$

, however, $s^*\mathcal{O}(1)$ is not trivial.

$\mathrm{Pic}(PGL(V)) \cong \mathbb{Z}/(\dim V)\mathbb{Z}$.

but $\mathcal{O}_{\mathbb{P}(V)}(\dim(V))$ will have eq. structure.

On the other hand $\mathcal{O}(1)$ is linearizable for action of $GL(V) \curvearrowright H^0(\mathcal{O}(1)) \cong V^*$.

if one restricts this action to $SL(V)$, we see that ~~this~~ induced action on $\mathcal{O}(\dim(V))$ is trivial on $\mu_{\dim(V)}$ hence descends to an action of $PGL(V)$.

Key facts (1) if $X_1 \rightrightarrows X_0$ are flat maps, then $\mathrm{QCoh}(X_\bullet)$ is Abelian, w/ kernels and cokernels formed on X_0.

(2) any coh. $\mathcal{F}$ admits $\mathcal{O}_X(-n)\otimes W \twoheadrightarrow \mathcal{F}$ where W is a G-repn. $\mathcal{O}_X(-n)\otimes\Gamma(X,\mathcal{F}(n)) \twoheadrightarrow \mathcal{F}$.

Thm if X is a normal proj. k-scheme, G connected linear gp. X has a G-action, then X is G-proj.

pf. Step 1. $G \circlearrowright \mathrm{Pic}(X/k)$ if $\mathcal{L}$ is fixed, then $\mathcal{L}^{\otimes n}$ for some $n \geq 1$ will be G-linearizable.

Step 2. if X is normal & proj. then components of $\mathrm{Pic}(X/k)$ is abelian variety, G is rat'l, so orbit maps are constant.

Prop Morita map $f: Y_\bullet \longrightarrow X_\bullet$

$\mathrm{Qcoh}(X_\bullet) \xrightarrow{f^*} \mathrm{Qcoh}(Y_\bullet)$ is an equivalence

pf

$$\begin{tikzcd}
 & & W_{22} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & W_{12} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & W_{02} \arrow[rr] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & & Y_2 \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
\mathrm{Qcoh}(X_\bullet) \arrow[r, "\cong"] & \mathrm{QCoh}(W_{\bullet\bullet}) & W_{21} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, shift left] \arrow[d, shift right] & W_{11} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & W_{01} \arrow[rr] \arrow[d, shift left] \arrow[d, shift right] & & Y_1 \arrow[d, shift left] \arrow[d, shift right] \\
 & \mathrm{QCoh}(Y_\bullet) \arrow[u, "\cong"] & W_{20} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & W_{10} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & W_{00} \arrow[r] \arrow[d] & Y_1 \arrow[r, "t"] \arrow[d, "s"] & Y_0 \arrow[d, dotted] \\
 & & X_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_1 \arrow[r, shift left] \arrow[r, shift right] & X_0 \arrow[r, "f_0"] & Y_0 \arrow[r, dashed] & \mathcal{X}
\end{tikzcd}$$

Defn. category fibered in gpoids: $F \xrightarrow{b} \mathrm{Sch}$, s.t.

① $\forall$ diagram

$$\begin{tikzcd}
\xi' \arrow[r, dotted] \arrow[d, dotted] & \xi \arrow[d, "b"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

$\exists$ "cartesian" arrow ξ' "formal pull back"

② fiber $F(U) := b^{-1}(U)$ is a gpoid.

Defn. a stack is a cat. fibered in gpoids, s.t. $\forall$ surjective étale map of schemes $U_0 \longrightarrow X$, the pullback is an equivalence of cat.: $F(X) \longrightarrow F(U_\bullet)$

Ex. (1) $[\cdot/G]$, the cat. F consists of pairs (X, E) where E/X is a principal G bundle, w/ maps G-eq.

(2) $G \circlearrowright Y$. Quotient stack $F = [Y/G] = \{X, E$ a principal G bundle$/X$, and a G-eq. map $E \longrightarrow Y\}$

- To any gpd scheme $X_\bullet$ associate cat. fibered in gpds: $\underline{X_\bullet} \longrightarrow Sch$.

  obj: $(U \text{ scheme}, \ \xi \in X_0(U))$.

  maps:
$$\begin{tikzcd} \xi' \arrow[r, "\gamma"] \arrow[d, mapsto, dash] & \xi \arrow[d, mapsto, dash] \\ U \arrow[r, "f"] & V \end{tikzcd} \qquad \gamma \in X_1(U), \ t(\gamma) = \xi', \ s(\gamma) = f^*(\xi)$$

- For an fibered gpoid F, $\exists$ canonical stackification $F \to F^a$. F^a is a stack, universal w.r.t. $F \longrightarrow$ stacks.

Ex. $\underline{(G \times X \rightrightarrows X)} \xrightarrow{\text{stackification}} [X/G]$.

$$\begin{tikzcd} (U \xrightarrow{f} X) \arrow[r, mapsto] & G \times U \arrow[r, "g \cdot f(u)"] \arrow[d, "\text{trivial } G\text{-bundle}"] & X \\ & U & \end{tikzcd}$$

Defn For
$$\begin{tikzcd} \mathcal{A} \arrow[rr] \arrow[dr] & & \mathcal{C}_2 \arrow[d, "f_2"] \\ & \mathcal{C}_1 \arrow[r, "f_1"] \arrow[ur, Leftrightarrow] & \mathcal{D} \end{tikzcd}$$
(homotopy) fiber product is universal diagram of this kind.

$\mathcal{C}_1 \times_{\mathcal{D}} \mathcal{C}_2$: obj $= (x \in \mathcal{C}_1, \ y \in \mathcal{C}_2, \ \text{iso}: f_1(x) \xrightarrow{\cong} f_2(y))$.

morphism $= \left(\begin{matrix} x_1 \to x_2 \\ y_1 \to y_2 \end{matrix} \right.$ which commutes w/ everything$)$.

Fact. for stacks $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}$. $\mathcal{X}_1 \times_{\mathcal{Y}} \mathcal{X}_2$ is still a stack.

2-Yoneda Lemma: $X \in Sch$, can regard as a cat. fibered in gpoids/Sch. $\underline{X} \rightrightarrows X$. obj $= (U, f: U \to X)$

mor $=$
$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}$$

Then $Map_{Sch}(\underline{X}, F) \xrightarrow{\cong} F(X)$ is equivalence as gpds.
So it justifies referring to a stack as <u>representable</u> if it's $\cong \underline{X}$. and representable map means...

Thm TFAE for a stack $\mathcal{X}$.

(1) $\mathcal{X} \cong (\underline{X_\bullet})^a$ for a <u>smooth</u> gpoid scheme (s, & t are smooth)

(2) $\mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times \mathcal{X}$ is representable by alg. spaces & $\exists$ smooth surjection $\underline{U} \longrightarrow \mathcal{X}$.

(3) $\exists$ <u>representable</u> smooth surjection $\underline{U} \longrightarrow \mathcal{X}$.

Furthermore, it's equivalent replacing smooth by fppf.

Defn Any stack satisfying these equivalence condition is <u>algebraic</u>

Rmk given $U \longrightarrow \mathcal{X}$, get a gpoid $U_0 = U \leftleftarrows U_1 = U_0 \times_{\mathcal{X}} U_0$

ex.
$$\begin{tikzcd} X \arrow[r] & X/G \\ G \times X \arrow[u] \arrow[r] \arrow[ur, phantom, "\lrcorner"] & X \arrow[u] \end{tikzcd}$$
is smooth & representable

homom: $\psi: G \to H$
$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ G \arrow[u, phantom, "\circlearrowleft"] \arrow[r, "\psi"] & H \arrow[u, phantom, "\circlearrowleft"] \end{tikzcd}$$
equivariant map.

$$\begin{tikzcd} G\times X \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "(\psi, f)"'] & X \arrow[d, "f"] \\ H\times Y \arrow[r, shift left] \arrow[r, shift right] & Y \end{tikzcd}$$

we get a map from $[X/G] \xrightarrow{[f]} [Y/H]$ and actually:

$$\begin{tikzcd} {[X/G]} \arrow[r, "{[f]}"] & {[Y/H]} \\ (H\times X)/G \arrow[u] \arrow[r] & Y \arrow[u] \end{tikzcd}$$

in particular, $[f]$ is representable by alg. space iff G acts freely on $H\times X$, which would be the case if $G \hookrightarrow H$

and then $X/G \cong (H \times^G X)/H$ $(= H\backslash(H\times X)/G)$.

$$\begin{tikzcd} X/G \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & Y/H \\ H\backslash(H\times X)/G \arrow[ur, "h\cdot f(x)"'] & \end{tikzcd}$$

Fact $X \to Y$, H equivariant, then
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X/H \arrow[r] & Y/H \end{tikzcd}$$

$$\begin{tikzcd} X/G \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & X/G \times X/G \arrow[d, phantom, "\cong" rotate=90] \\ G\times X/G\times G \arrow[r] & X\times X/G\times G \end{tikzcd}$$

$(g_1, g_2)\cdot(g, x) = (g_1 g g_2^{-1}, g_2 x)$.

consider $X/G = \mathcal{X}$ w/ X a G-q-proj.

$(G\times G)\times^G X/G\times G \cong \mathcal{X} \xrightarrow{\Delta} \mathcal{X}\times\mathcal{X}$.

This diagonal is presented by $G\times G$-eq. map $G\times X \to X\times X$.

Claim: $\mathcal{X} \xrightarrow{\Delta} \mathcal{X}\times\mathcal{X}$ is affine.

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X}\times\mathcal{X} \\ G\times X \arrow[u] \arrow[r, "\text{orbit}"] & X\times X \arrow[u] \\ Y \arrow[u] \arrow[r] & \mathrm{Spec}(R) \arrow[u, "f"'] \end{tikzcd}
\qquad
\begin{tikzcd} G\times \mathrm{Spec}(R) \arrow[r] & \mathrm{Spec}(R)\times X \\ Y \arrow[u] \arrow[r] & \mathrm{Spec}(R) \arrow[u, "\xi\text{, closed}"'] \end{tikzcd}$$

Y is affine b/c X is sep'd & G is affine.

Defn. geometric stack is a q-cpt alg. stack w/ affine diagonal.

e.g. if $\mathcal{X} = */G$, ξ_1, ξ_2 are principal G-bundles ~~then~~ on T corresponds to $T \to \mathcal{X}$, then pull back is just $\underline{\mathrm{Isom}}_T(\xi_1, \xi_2)$ which is again a principal G-bundle on T, in particular it's affine.

"linear" objects tend to have affine automorphism gps.

Ex. X/k is a proper scheme. $\underline{\mathrm{Coh}}_{X/k}$: open $U \mapsto \{U\text{-flat families of coh. sheaves on } X\times U\}/\cong$

Fact: $\underline{\mathrm{Coh}}_{X/k}$ has affine diagonal. $\underline{\mathrm{Bun}}_{X/k}$: . . . v.b.

$\underline{\mathrm{Pic}}_{X/k}$: . . . l.b.

Ex. consider $\underline{Bun}_{C/k}$ for C a smooth curve$/k$, then connected components are given by rk & deg.

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^2 \to \mathcal{O}(1) \to 0$$

hence $\mathcal{O}^2 \rightsquigarrow \mathcal{O}(-1) \oplus \mathcal{O}(1)$ on $\mathbb{A}^1$.

Ex. not all alg. stacks are geometric:
stacks of flat families of curves w/ geom. conn. reduced fiber w/ arithmetic genus $=1$. (automorphism contains elliptic curves, hence not affine).

Thm (Totaro) $\mathcal{X}$ is a geometric stack, if $\mathcal{X}$ is normal, TFAE.
① every coh. sheaf is a quotient of v.b. (resolution property).
② $\mathcal{X} = X/GL_n$ for some q-affine scheme X over k, $\Leftrightarrow$ ③ $\mathcal{X} = \mathrm{Spec}(R)/G \leftarrow$ some affine gp

Ex. on $\mathcal{X} = X/G$, where ~~G~~ X is G-proj, we saw resolution property holds true, & $\mathcal{X} = (\mathrm{Cone}(X) - \{0\})/GL_1$!

GIT Alper: Good Moduli Space

Lemma $\mathcal{X}$ geometric stack, then every $F \in QCoh(\mathcal{X})$ is a union of its coherent sub-sheaves

Because geometric stack has a presentation $X_\bullet$ where X_0, X_1 both affine.

Even more $QCoh(\mathcal{X}) = \mathrm{Ind}(Coh(\mathcal{X}))$. meaning
(1) $\mathrm{Hom}(Coh, -)$ commutes w/ filtered colimit.
(2) $QCoh(\mathcal{X}) \longrightarrow \mathrm{Fun}(Coh(\mathcal{X})^{op}, Ab)$ is an equivalence.

Fact For any alg. stack $\mathcal{X}$, $QCoh(\mathcal{X})$ is a Grothendieck cat.

Defn G. cat has arbitrary direct sums, filtered colimits, filtered colimits are exact and $\exists$ a generating obj U, i.e., $\forall M \subsetneq N$, $\exists$ a map $U \to N$ which doesn't factor thru M.

Thm In a Grothendieck cat., $\exists$ enough injectives, (and has enough K-injective cplxes).

Any map of stacks $f: \mathcal{X} \longrightarrow \mathcal{Y}$ can be modeled as map of gpoid

$$\begin{tikzcd}
\mathcal{X} \arrow[r] & \mathcal{Y} \\
V_0 \arrow[r] \arrow[u] & U_0 \arrow[u] \\
V_1 \arrow[r] \arrow[u, shift left] \arrow[u, shift right] & U_1 \arrow[u, shift left] \arrow[u, shift right]
\end{tikzcd}$$

$\Rightarrow \exists$ a pullback functor $f^*: QCoh(\mathcal{Y}) \longrightarrow QCoh(\mathcal{X})$.

one can define pushforward $f_* : QCoh(\mathcal{X}) \longrightarrow QCoh(\mathcal{Y})$ as the right adjoint of f^*.

Ex:

$$\begin{tikzcd} X/G \arrow[r, "f"] & Y \\ X \arrow[u] \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

given $E \in QCoh(X/G)$. $\tilde{f}(E|_X) \in QCoh(Y)$ actually canonically lives in $QCoh(Y \times (\cdot/G))$

$$\begin{tikzcd} X/G \arrow[r, "p"] \arrow[rr, bend left, "f"] & Y \times (\cdot/G) \arrow[r, "q"] & Y \\ X \arrow[u] \arrow[r, "p'"] & Y \arrow[u] & \end{tikzcd}$$

(1) $p_*(E)$ arises from $p'_*(E|_X)$ via smooth descent.

(2) q_* is taking invariants.

Rmk if G is linearly reductive, namely, $(-)^G$ is exact, and if $Y = pt$, then $R\Gamma(X/G, E)^G = Rf_*(E)$.

Defn $q: \mathcal{X} \to Y$, $\mathcal{X}$ alg. ~~space~~ stack, Y alg. space, say q is a good moduli space if

(1) $q_* : QCoh(\mathcal{X}) \to QCoh(Y)$ is exact.

(2) $\mathcal{O}_Y \to q_* \mathcal{O}_{\mathcal{X}}$ is an isom.

Main example G linearly reductive, $X = Spec(R)$, then

$$\begin{tikzcd} Spec(R)/G \arrow[r] \arrow[d, "\text{representable affine}"'] & Spec(R^G) \\ Spec(R^G) \times (\cdot/G) \arrow[ur] & \end{tikzcd}$$

is a good moduli space.

Main properties of GMS: if $q: \mathcal{X} \to Y$ is GMS, then

(1) q is surjective, universally closed, universally submersive

(2) k alg. closed & $x_1, x_2 \in \mathcal{X}(k)$, then

$q(x_1) = q(x_2)$ iff $\overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \phi$ in $\mathcal{X} \times_Y Spec(k)$

(3) stable under base change along $Y' \to Y$ and fpqc local on Y.

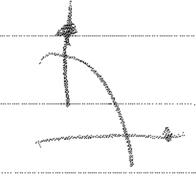
(4) if $\mathcal{X}$ is locally Noetherian, then Y is locally Noetherian; finite type $/k \Rightarrow$ f.t. $/k$.

Ex. $\mathbb{C}^2 = \mathbb{C}(1) \oplus \mathbb{C}(-1)$. $R^G = \mathbb{C}[xy]$

$\mathbb{C}^* \circlearrowright \mathbb{C}^2$

$$\mathbb{C}^2 \longrightarrow \mathbb{C}$$
$$(x, y) \longmapsto xy.$$

Nonex blow up $\mathbb{C}^2/\mathbb{C}^*$ at origin $\cong Tot_{\mathbb{P}^1}(\mathcal{O}(-1))$



Prop. Given a Cartesian diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f'"] \arrow[d, "q'"] & \mathcal{X} \arrow[d, "q"] \\ Y' \arrow[r, "f"] & Y \end{tikzcd}$$

(a) if f is fpqc & q' is a GMS, then q is a GMS.

(b) for f arbitrary, q is a GMS $\Rightarrow$ q' is a GMS.

pf. (a) follows from flat base change & permanence under fp base change.

(b) use (a) to devissage to the case where f^0 is quasi-affine ...

Key fact if $q: \mathcal{X} \to Y$ is a GMS, then $F \to q_* q^* F$ is an isom.

pf ~~idea~~ (1) reduce to Y is affine by (a) of Prop. above

(2)
$$\begin{tikzcd} \mathcal{O}_Y^{\oplus I} \arrow[r] \arrow[dd, bend right] & \mathcal{O}_Y^{\oplus J} \arrow[r] \arrow[dd, bend left] & F \arrow[r] \arrow[dd, bend right, "\cong"] & 0 \\ \cong\ \mathcal{O}_{\mathcal{X}}^{\oplus I} \arrow[r] & \mathcal{O}_{\mathcal{X}}^{\oplus J} \arrow[r, "\cong"] & q^*(F) \arrow[r] & 0 \\ q_* \mathcal{O}_{\mathcal{X}}^{\oplus I} \arrow[r] & q_* \mathcal{O}_{\mathcal{X}}^{\oplus J} \arrow[r] & q_* q^*(F) \arrow[r] & 0 \end{tikzcd}$$

Consequences (1) if $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}}$ is an ideal sheaf for closed substack

$$q_*(\mathcal{O}_{\mathcal{X}}/\mathcal{I}) \cong \mathcal{O}_Y / q_* \mathcal{I}$$

(2) for ideal sheaves $\mathcal{I}_1$ & $\mathcal{I}_2$

$$q_*(\mathcal{I}_1) + q_*(\mathcal{I}_2) = q_*(\mathcal{I}_1 + \mathcal{I}_2)$$

(3) $J \subseteq \mathcal{O}_Y$ is an ideal sheaf and $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}}$ is the preimage ideal sheaf, then $J \xrightarrow{\cong} q_* \mathcal{I}$

These have geometric consequences:

(2) $\Rightarrow$ $\mathcal{Z}_1, \mathcal{Z}_2 \subseteq \mathcal{X}$ closed substacks, then schematic image $q(\mathcal{Z}_1) \cap q(\mathcal{Z}_2) = q(\mathcal{Z}_1 \cap \mathcal{Z}_2)$

which leads to
$$\begin{tikzcd} \mathcal{X} \arrow[d, "q"] & \operatorname{Spec}(\bar{k}) \arrow[l, "x_1, x_2"'] \\ Y \end{tikzcd}$$

$q(x_1) \cap q(x_2) \neq \phi \iff \overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \phi$ ..

(3) $\Rightarrow$ if $\mathcal{X}$ is Noetherian, then so is Y

ascending chain of ideal sheaves on Y.

$J_1 \subseteq J_2 \subseteq \cdots$, preimage $\mathcal{I}_1 \subseteq \cdots$ stabilizes

then $J_\cdot = q_*(\mathcal{I}_\cdot)$ also stabilizes.

Cor (Hilbert 14) if R is f.g. G-equivariant k-alg., and G is linearly reductive then R^G is f.g.

pf. reduce to $R = k[V]$ for some rep V.

$\operatorname{Spec}(k[V]^G) \longleftarrow \operatorname{Spec}(k[V])/G$ is a GMS

$\Rightarrow$ $k[V]^G$ is Noetherian.

Lemma a graded ring $A = k \oplus \bigoplus_{n>0} A_n$ is f.g.$/k$ iff it's Noetherian.

Idea for finding GMS

cover $\mathcal{X}$ by open substacks which have GMS, then glue

this means finding open affines in X which are G-equivariant.

Can maybe construct GMS for an open ~~subscheme~~ substack of $\mathcal{X}$

Ex. $\mathrm{Spec}(R)/G$ can find many open substacks w/ GMS

- $f \in R^G$, then $\{f \neq 0\} \subseteq \mathrm{Spec}(R)$ is G-equiv.
- consider a character $\chi : G \to \mathbb{G}_m$, and $f \in R$ s.t. $g(f) = \chi(g) \cdot f$, then $\{f \neq 0\} \subseteq \mathrm{Spec}(R)$ is also G-equivariant & affine. f is called semi-invariant.
- Given $\chi : G \to \mathbb{G}_m$, can define $\mathrm{Spec}(R)^{\chi\text{-s.s.}} \subseteq \mathrm{Spec}(R)$

$$\mathrm{Spec}(R)^{\chi\text{-s.s.}} = \left\{ x \in \mathrm{Spec}(R) \;\middle|\; \begin{array}{l} \exists\, \chi^n \text{ semi-inv } f \text{ w/ } f(x) \neq 0 \\ \text{for some } n > 0 \end{array} \right\}$$

$$= \bigcup_{\substack{f \text{ semi-inv.} \\ \text{of } \chi^n}} \mathrm{Spec}(R[f^{-1}])$$

$$= \bigoplus_{n \geq 0} (R \otimes k(\chi^n))^G$$

$\mathrm{Spec}(R)^{\chi\text{-s.s.}}/G$ has GMS $= \mathrm{proj}$ (ring of χ^n-semi-inv)

$$\downarrow$$

$$\mathrm{Spec}(R^G)$$

$$\begin{tikzcd} \mathbb{C}^n/\mathbb{C}^* \arrow[d] & \supset & \mathbb{C}^n - \{0\}/\mathbb{C}^* \arrow[d, "\cong"] \\ \mathbb{P}^0 & & \mathbb{P}^{n-1} \arrow[ll, "\mathrm{proj}"'] \end{tikzcd}$$

Defn $\mathcal{X}$ is a tame moduli space (geometric quotient if $\mathcal{X} = X/G$) if $\mathcal{X} \xrightarrow{\varphi} Y$ is a GMS and a bijection on geometric points: $\pi_0(\mathcal{X}(\bar{k})) \xrightarrow{\cong} Y(\bar{k})$.

E.g. if $\mathcal{X}$ is a nice enough Deligne-Mumford stack ($\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is finite), then $\exists$ tame moduli spaces

If $\mathcal{X}$ is an alg. stack, define $x \in \mathcal{X}$ is <u>prestable</u> if $\exists$ open substack $x \in \mathcal{U} \subseteq \mathcal{X}$ which is cohomologically affine and geometric pts are closed (orbits are closed)

$\mathcal{X}$ is cohomologically affine means $\mathcal{X} \to \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a GMS

$\mathcal{X}^s_{pre} = \{$ prestable pts in $\mathcal{X} \}$.

Prop. if $\mathcal{X}$ is q.c. alg. stack, then $\mathcal{X}^s_{pre}$ has a tame moduli space

E.g. $\mathbb{C}^2 = \mathbb{C}(1) \oplus \mathbb{C}(-1) \circlearrowleft \mathbb{C}^*$

$(\mathbb{C}^2/\mathbb{C}^*)^s_{pre} = \{x \neq 0\} \cup \{y \neq 0\}/\mathbb{C}^* =$ affine line w/ double origin

$\mathcal{X} = \mathrm{Spec}(R)/G \quad \chi : G \to \mathbb{G}_m$

$\{x \in \mathrm{Spec}(R) \mid \exists\, \chi^n\text{-semi-inv. } f \text{ s.t. } f(x) \neq 0\} \longrightarrow \mathrm{Proj}\left(\bigoplus_{n \geq 0} (R(\chi^n))^G\right)$

is a GMS.

Rmk. if

$$\begin{tikzcd} U \arrow[r, phantom, "\overset{\text{open}}{\subseteq}"] \arrow[d, "\mathrm{GMS}"'] & \mathcal{X} \arrow[d, "\mathrm{GMS}"] \\ V \arrow[r, "\exists!"] & Y \end{tikzcd}$$

might not be open.

In general: $R(\mathcal{X})$ is invertible G-equivariant R-module $\mathrm{Spec}(R)/G \longrightarrow */\mathbb{G}_m$, in fact $R(\mathcal{X})$ is an invertible sheaf $\mathcal{L}$ on $\mathcal{X} = \mathrm{Spec}(R)/G$

Generalization is associating any $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$, $\bigoplus_{n\geq 0}\Gamma(\mathcal{X}, \mathcal{L}^n)$

In general, get a map $\mathcal{X}^{s.s.}(\mathcal{L}) \overset{\text{def}}{=} \left\{ x \in \mathcal{X} \,\middle|\, \begin{array}{l} \exists f \in \Gamma(\mathcal{X}, \mathcal{L}^n) \text{ s.t. } f(x) \neq 0 \text{ and} \\ \mathcal{X}_f \text{ is coh. affine} \end{array} \right\}$

$$\downarrow q$$

$$\mathrm{Proj}\Big(\bigoplus_{n\geq 0}\Gamma(\mathcal{L}^n)\Big)$$

Thm ($\mathcal{X}$ is f.t./k)

(1) image of q is open and $q: \mathcal{X}^{s.s.}(\mathcal{L}) \longrightarrow Y = \mathrm{Im}(q)$ is a GMS

(2) $\exists$ ample line bundle M on Y s.t. $q^*(M) = \mathcal{L}^N$ for some N, Y is quasi-proj.

(3) $\exists$ a "stable" locus $\mathcal{X}^s(\mathcal{L}) \subseteq \mathcal{X}^{s.s.}(\mathcal{L})$

$\left\{ x \in \mathcal{X} \,\middle|\, \begin{array}{l} x \in \mathcal{X}_f,\ \mathcal{X}_f \text{ is coh. affine} \\ \text{and geometric pts are closed} \end{array} \right\}$

then $\exists$ open subspace $Y^s \subseteq Y$ which is a tame moduli space for $\mathcal{X}^s(\mathcal{L}) \subseteq \mathcal{X}^{ss}(\mathcal{L})$

Analysis of stability

Ex. $\mathrm{Spec}(k[t])/\mathbb{G}_m$, t has wt -1 $\quad k[t]\cdot e \leftarrow$ wt n

$\mathcal{L}$ must be of the form $\mathcal{O}_{\mathbb{A}^1}(n)$, $\mathbb{G}_m \xrightarrow{z} \mathbb{G}_m$

$\{1\}$ is semistable iff $\mathrm{wt}_{\mathbb{G}_m}(\mathcal{L}|_{\{0\}}) \geq 0$

Observation if $x \in X = \mathrm{Spec}(R)$, $\lambda: \mathbb{G}_m \to G$, if $\lambda(t)\cdot x: \mathbb{A}^1 - \{0\} \to X$ extends to $\mathbb{A}^1$, then it extends uniquely b/c X is separated

$\Rightarrow \mathbb{A}^1 \longrightarrow X$ equivariant w.r.t. $\mathbb{G}_m \to G$.

$\Rightarrow \mathbb{A}^1/\mathbb{G}_m \xrightarrow{\varphi} X/G$

Now, if $x \in X^{s.s.}(\mathcal{L})$, then $\exists f \in \Gamma(X, \mathcal{L}^n)^G$, $f(x) \neq 0$

$\Rightarrow \exists f \in \Gamma(\mathbb{A}^1/\mathbb{G}_m, \varphi^*(\mathcal{L}^n))$ w/ $f(1) \neq 0$.

$\Rightarrow \mathrm{wt}_{\mathbb{G}_m}(\mathcal{L}|_{\{\varphi(0)\}}) \geq 0$.

Defn. given (x, λ) s.t. $\lim_{t\to 0} \lambda(t)\cdot x = y$ exists, $\mu(x, \lambda) = \mathrm{wt}_\lambda(\mathcal{L}|_{\{y\}})$

Thm (Hilbert–Mumford)

(1) $x \in X^{s.s.}(\mathcal{L}) \Leftrightarrow \mu(x, \lambda) \geq 0$ $\forall \lambda$, s.t. $\lim_{t\to 0}\lambda(t)\cdot x$ exists

(2) $x \in X^s(\mathcal{L}) \Leftrightarrow \mu(x, \lambda) > 0$ —— " ——

pf. step 1: reformulate s.s.: let $V = \underline{\mathrm{Spec}}(\bigoplus_n \mathcal{L}^n) = \mathrm{Tot}_X(\mathcal{L}^\vee) \to X$

claim: $x \in X$ is $\mathcal{L}$-s.s. iff $\exists x^* \in V \setminus X$ over x, s.t. $\overline{\{G\cdot x^*\}} \cap X = \phi$

step 2: $\overline{\{G\cdot x^*\}} \cap X \neq \phi$ iff $\exists \lambda: \mathbb{G}_m \to G$ s.t. $\lim_{t\to 0}\lambda(t)\cdot x^* \in X$

Step 3: λ ~~has~~ satisfies such condition $\Leftrightarrow \mu(x, \lambda) < 0$.

if $\overline{\{G\cdot x^*\}} \cap X \neq \emptyset$, then $\exists$ following diagram

$$\begin{tikzcd} & G \arrow[dr] & \\ \mathrm{Spec}(k((t))) \arrow[ur] \arrow[rr] \arrow[d] & & \{G\cdot x^*\} \arrow[d] \\ \mathrm{Spec}(k[[t]]) \arrow[rr] & & \overline{\{G\cdot x^*\}} \end{tikzcd}$$

$0 \longmapsto$ pt in 0 section

Iwahori: $G(k[[t]])\backslash G(k((t)))/G(k[[t]]) \underset{set}{\cong} \mathrm{Hom}(\mathbb{G}_m, G)/\underset{conj.}{G}$
for any reductive G.

Short version of Main thm in GIT: $\mathcal{X} \xrightarrow{q} Y$ is a finite type GMS of a geometric stack, Y qcqs alg. space $/ k=\bar{k}$, char. 0.

Let $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$, then

(1) say $x \in \mathcal{X}(k)$ is semistable if $\forall f: \mathbb{A}^1/\mathbb{G}_m \to \mathcal{X}$
$\mathrm{wt}_{\{0\}}(f^*\mathcal{L}) \geq 0$.

(2) $\mathcal{X}^{ss}(\mathcal{L})$ is open and $\mathcal{X}^{s.s.}(\mathcal{L}) \longrightarrow \mathrm{Proj}_Y\left(\bigoplus_{n\geq 0} q_*(\mathcal{L}^n)\right)$ is a GMS and $\mathcal{L}^N$ descends to a ~~relatively am~~ relatively ample line bundle for $N >> 0$.

Thm: $\mathcal{X}$ is a f.t. geometric stack and $\mathcal{X} \xrightarrow{q} Y$ is a GMS then $\exists$ a cartesian diagram of the form (w/ G linearly red.)

$$\begin{tikzcd} \mathrm{Spec}(R)/G \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "q"] \\ \mathrm{Spec}(R^G) \arrow[r, "\text{étale}", "\text{surj.}"'] & Y \end{tikzcd}$$

Prop: $\mathrm{Map}(\mathbb{A}^1/\mathbb{G}_m, X/G)/_{\mathrm{isom.}} \cong \left\{ \begin{array}{l} \text{pairs } (\lambda, x) \\ \text{s.t. } \lim_{t\to 0} \lambda(t)\cdot x \\ \text{exists} \end{array} \right\} \Big/ \begin{array}{l}(\lambda,x) \sim (\lambda, px) \\ \sim (g\lambda g^{-1}, gx)\end{array}$

for $g \in G$, $p \in P_\lambda$

where $\{g \in G, \text{ s.t. } \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists}\} = P_\lambda = \left(\begin{array}{l} \text{block upper triangular w/r/t} \\ \text{eigenspaces of } \lambda: \mathbb{G}_m \to G \end{array} \right)$
for some embedding $G \hookrightarrow GL_n$.

Consequences of H-M criterion:

(1) $X^{s.s.}(\mathcal{L}) = X^{s.s.}(\mathcal{L}^n)$, $n>0$. So one can consider GIT for any G-linearized ample $\mathcal{L}$, and stability is well-defined w/r/t $\mathcal{L} \in \mathrm{Pic}(X/G) \otimes_{\mathbb{Z}} \mathbb{Q}$

(2) $X^{s.s.}(\mathcal{L})$ only depends on $c_1(\mathcal{L}) \in H^2_G(X)$, b/c weight of $f^*(\mathcal{L})|_{\{0\}}$ only depends on $c_1(f^*(\mathcal{L})) = f^* c_1(\mathcal{L}) \in H^2_{\mathbb{G}_m}(\mathbb{A}^1)$.

(3) perturbation of stability
how does $X^{s.s.}(\mathcal{L} + \varepsilon \mathcal{L}')$ compare to $X^{s.s.}(\mathcal{L})$ for $0 < \underset{\in \mathbb{Q}}{\varepsilon} << 1$

Answer: $X^{s.s.}(\mathcal{L} + \varepsilon\mathcal{L}') \subseteq X^{s.s.}(\mathcal{L})$

informal idea: for unstable pt of $\mathcal{L}$, we have $\mathrm{wt}(f^*(\mathcal{L})|_{\{0\}}) < 0$, so for small enough ε, we get $\mathrm{wt}(f^*(\mathcal{L} + \varepsilon\mathcal{L}')) < 0$

(4) if $Y/G \xrightarrow{\pi} X/G$ representable finite map, then $Y^{s.s.}(\pi^*(\mathcal{L})) = \pi^{-1}(X^{ss}(\mathcal{L}))$.

$$\begin{tikzcd} \mathbb{G}_m/\mathbb{G}_m \arrow[r] \arrow[d] & Y/G \arrow[d] \\ \mathbb{A}^1/\mathbb{G}_m \arrow[r] & X/G \end{tikzcd}$$

$y \in \pi^{-1}(x)$

$1 \longmapsto x$

E.g. $Y = (\mathbb{P}^1)^n \circlearrowleft SL_2$ $\quad \mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(r_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^1}(r_n)$

all $\lambda: \mathbb{G}_m \to SL_2$ are conjugate to $\begin{pmatrix} t^n & \\ & t^{-n} \end{pmatrix}$

$\Leftrightarrow$ choosing coordinate system on $\mathbb{P}^1$.

$y = (l_1, \ldots, l_n) \in Y$. limit pt of $[\alpha : \beta]$ as $t \to 0$ $= \begin{cases} [0:1] & \text{if } \beta \neq 0 \\ [1:0] & \text{if } \beta = 0 \end{cases}$

$\mathrm{wt}_\lambda \mathcal{O}_{\mathbb{P}^1}(r_i)\big|_{[0:1]} = r_i \quad \big|_{[1:0]} = -r_i$.

So $\mathrm{wt} \geq 0$ iff $\sum_{l_i = [1:0]} r_i \leq \sum_{l_i \neq [1:0]} r_i$

$y = (l_1, \ldots, l_n)$ is $\mathcal{L}$-semistable $\Leftrightarrow \forall l \in \mathbb{P}^1, \sum_{l_i = l} r_i \leq \sum_{l_i \neq l} r_i$

$SL_2 \circlearrowright \mathbb{P}(\mathrm{Sym}^n(k^2))$ $\quad \phi(x,y) \in \mathrm{Sym}^n(k^2)$ is semistable iff $\nexists$ linear factors of multiplicity $> \frac{n}{2}$.

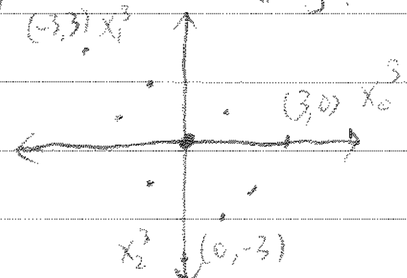
(5) fix $T \subseteq G$ max'l torus.

$(x \in X \circlearrowleft G$ is semistable$) \Leftrightarrow (\forall g \in G,\ g \cdot x$ is T-semistable$)$.

E.g. $SL_3 \circlearrowright \mathbb{P}(\mathrm{Sym}^3(\mathbb{C}^3)) = \{\deg 3 \text{ curves in } \mathbb{P}^2\}$.

$\mathbb{G}_m^2 = T \hookrightarrow SL_3$

$\left\{ \begin{pmatrix} t_1 & & \\ & t_1^{-1} t_2 & \\ & & t_2^{-1} \end{pmatrix} \right\}$



$\lim_{t \to 0} \lambda(t) \cdot P$ = Projection of P onto lowest wt eigenspace in which P has nonzero coefficients.

P is semistable $\Leftrightarrow$ $\mathrm{st}(P)$ contains origin.

fixing an W-invariant inner product $|\cdot|$, $|\lambda|$ well defined $\forall \lambda: \mathbb{G}_m \to G$ (by conjugating into T)

Given $\mathrm{St}(p) \subseteq M_{\mathbb{R}}$, $\exists! \lambda \in N_{\mathbb{R}}$ w/ $|\lambda| = 1$, which minimizes $\nu(p, \lambda) = \frac{1}{|\lambda|} \mu(p, \lambda) = \max_{\chi \in \mathrm{St}(p)} \langle \frac{\lambda}{|\lambda|}, \chi \rangle$

the function $\max_{\chi \in \mathrm{St}(p)} \langle -, \chi \rangle$ is strictly convex upward on unit sphere in $N_{\mathbb{R}}$, where M = char. lattice, N = cochar. lattice.

Conclusion $\exists! \lambda \in N_{\mathbb{R}}$ which minimizes $\nu(p, \lambda) = \frac{1}{|\lambda|} \mathrm{wt}_\lambda(\mathcal{O}(1)_{p_0})$ and this λ only depends on $|\cdot|$ and $\mathcal{O}(1)$, not on choice of $T \subseteq G$.

Kempf $\exists \lambda$, unique up to conjugation by $g \in P_\lambda$, which minimizes $\nu(p, \lambda)$ for any unstable P.

Thm (Kempf). Let X be proj. or affine vty $/k$ w/ linearized G action, G reductive, $\mathcal{L} \in NS_G(X)$. Fix W-invariant inner product on $N = \mathrm{cochar}(T)$. Assume $\mathcal{L}|_X$ is NEF, then

(a) $\forall P \in X^{us}$, $\exists$ unique map $f: \mathbb{A}^1/\mathbb{G}_m \to X/G$ with $f(1) \cong P$, which minimizes $\nu(f) = \nu(P, \lambda)$ up to $(p, \lambda) \mapsto (p, \lambda^n)$.

(b) if P specializes to q, then $M(q) \leq M(p)$, where $M(p) := \min_{f} \nu(p, \lambda)$.

(c) up to conjugation, only finitely many λ appear as optimal destabilizers in (a).

assume X is affine

idea: use the spherical building of G, $Sph(G)$:

(1) $\forall$ max'l tori $T \subseteq G$, let S_T denote unit sphere in $\mathrm{cochar}(T)_{\mathbb{R}}$.

(2) $\forall$ Borel subgp $T \subseteq B \subseteq G$, get a top dim'l cone (Weyl chamber) in $\mathrm{cochar}(T)_{\mathbb{R}}$, intersecting w/ unit sphere get a polyhedral sector $\Delta_B \subseteq S_T$

(3) Glue S_T to $S_{T'}$ along Δ_B, $\forall B \supseteq T, T'$

then we consider $Deg(P) \subseteq Sph(G)$ consisting of pts of λ for which $\lim\limits_{t\to 0} \lambda(t) \cdot P$ exists.

$$Deg(P) \cap S_T \subseteq S_T \subseteq Sph(G)$$

$\quad \| \quad$ convex ~~poly~~ polyhedron inside S_T.

Kempf's Q: consider $\nu : Deg(P) \longrightarrow \mathbb{R}$

$$\lambda \longmapsto \frac{1}{|\lambda|} wt_\lambda \left(\mathcal{L}|_{\lim\limits_{t\to 0} \lambda(t)\cdot P}\right)$$

(Rmk: restricted to S_T, this function is of the form:

$\nu = \frac{1}{|\lambda|} \max\limits_{l_i} \langle -\lambda, l_i \rangle$ for finitely many l_i.

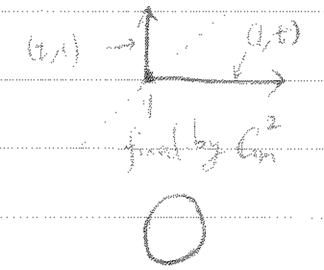
Q: $\exists!$ minimizer for ν on $Deg(P)$?

Answer is yes.

existence: consider poset of (tori $T \subseteq T_{max}$ (fixed) $\subseteq G$, along w/ a choice of conn'd component $Z \subseteq X^T$). Let $\mathcal{C}$ be the union of spheres $(T, Z \subseteq X^T) \rightsquigarrow$ unit spheres in $\mathrm{cochar}(T)_{\mathbb{R}}$ if $T \subseteq T'$ & $Z' \subseteq Z$, then glue $S_{T,Z}$ to $S_{T',Z'}$ along natural inclusion. $/ W(G)$

$w \cdot (Z, T) = (w \cdot Z, wTw^{-1})$.

e.g. $\mathbb{A}^2 / \mathbb{G}_m^2$



$\mathbb{A}^2/GL_2$ quotient out reflection along dotted line

Observation: (1) $\mathcal{C}$ is cpt.

(2) $\exists$ continuous map $Deg(P) \longrightarrow \mathcal{C}$

$$\lambda \longmapsto (P_0, \lambda) \cdot \left(\begin{array}{c}\text{conjugacy class} \\ \text{in } T_{max}\end{array}\right)$$

then consider $Z \ni P_0$ conn'd component of $X^{Stab_{T_{max}}(P_0)}$ have sphere $S_{(Z, Stab_{T_{max}}(P_0))}$.

the function ν is induced via this map by a continuous fctn $\mathcal{C} \xrightarrow{\nu} \mathbb{R}$.

On each $S_{(Z,T)}$ ν is $\nu(\lambda) = \frac{1}{|\lambda|} wt_\lambda \mathcal{L}|_Z$.

$\Rightarrow \exists$ minimizer of ν b/c $Deg(\lambda) \to \mathcal{C}$ is closed.

Uniqueness: say you have a homomorphism $\phi : \mathbb{G}_m^2 \longrightarrow G$ w/ finite kernel and $P \in X$ s.t. $\lim\limits_{t\to 0} \phi(t^a, t^b) P$ exists $\forall\, a, b \geq 0$

So we get
$$\begin{array}{ccc} \mathbb{A}^2 & \longrightarrow & X \\ \circlearrowleft & (1,1) \longmapsto P & \circlearrowleft \\ \mathbb{G}_m & \longrightarrow & G \end{array}$$

think of this as a family of pts in $\mathrm{Deg}(P) \rightsquigarrow (t^a, t^b)$ defines a line segment in $S_T \subseteq \mathrm{Sph}(G)$ which is contained in $\mathrm{Deg}(P)$. restrict ν to this line segment, then it's of the form $\nu(\lambda) = \frac{1}{|\lambda|}\langle -\lambda, l \rangle$ for some $l \in \mathrm{Char.}(\mathbb{G}_m^2)_{\mathbb{R}}$.

Lemma such a fctn is strictly convex upward!

if $p \rightsquigarrow q$ then $M(q) \leq M(p)$. $\forall \lambda : \mathbb{G}_m \to G$

B-B strata $= Y_\lambda / P_\lambda \xrightarrow{i} X/G$ is proper

$\xrightarrow[\text{immersion}]{\text{closed}} X/P_\lambda \longrightarrow$ rep'ble, G/P_λ-bdle

Lemma. assume $k = \bar{k}$ ($\exists$ modification for general k).
choose a set of representations of conjugacy classes of 1-param. gps. $\lambda : \mathbb{G}_m \longrightarrow G$, then $\exists$ bijections

k-pts of $\bigsqcup_{[\lambda]} Y_\lambda / P_\lambda \xleftrightarrow{(1)}$ eq. classes of test data (X, λ)

$\xleftrightarrow{(2)}$ isom. classes of $\mathbb{A}^1/\mathbb{G}_m \longrightarrow X/G$

(1) $[y] \in Y_\lambda / P_\lambda \longmapsto (y, \lambda)$

(2) • works in family $\bigsqcup_{[\lambda]} Y_\lambda / P_\lambda \cong \underline{\mathrm{Map}}(\mathbb{A}^1/\mathbb{G}_m, X/G)$

a T-pt of RHS is a map $T \times \mathbb{A}^1/\mathbb{G}_m \longrightarrow X/G$.

• any G-bundle on $\mathbb{A}^1/\mathbb{G}_m$ is isom. to $E_\lambda = {}_{\mathbb{G}_m}\mathbb{A}^1 \times G \circlearrowleft G$ right mult for some λ : $\mathbb{G}_m$ $t \cdot (z, g) = (t \cdot z, \lambda(t) \cdot g)$.

$(\mathbb{A}^1 \times G)/\mathbb{G}_m \longrightarrow \mathbb{A}^1/\mathbb{G}_m$.

Kempf stratification

$\exists$ a finite list of 1-param. subgps $\lambda_1, \ldots, \lambda_n$ which are minimal destabilizers for unstable pts of X (up to conjugation)

$$\begin{array}{ccc} \text{consider} \quad Y_{Z_\alpha, \lambda_\alpha} & \xrightarrow[\text{closed}]{i_\alpha} & X \\ (y \mapsto \lim_{t \to 0} \lambda(t) y) = \pi_\alpha \downarrow & & \text{when } X \text{ is affine} \\ Z_\alpha & & \end{array}$$

$$\begin{array}{ccc} \text{Gives} \quad Y_{Z_\alpha, \lambda_\alpha} / P_{\lambda_\alpha} & \xrightarrow[\text{proper}]{i_\alpha} & X/G \\ \downarrow \pi_\alpha & \text{\& rep'ble} & \\ Z_\alpha / L_\alpha & \text{where} & L_\alpha = (\text{centralizers of } \lambda). \end{array}$$

Kempf/Kirwan (1) let $Z_\alpha^{s.s.}$ denote open complements of $\mathrm{im}(i_\beta) \cap Z_\alpha$ for $\beta > \alpha$, and $Y_\alpha^{s.s.} = \pi_\alpha^{-1}(Z_\alpha^{s.s.})$, then
$Y_\alpha^{s.s.}/P_\alpha \hookrightarrow X/G$ is a local immersion whose closure lies in $\bigcup_{\beta > \alpha} \mathrm{im}(i_\beta)$.
$(\beta \geq \alpha \Leftrightarrow \nu(Z_\beta, \lambda_\beta) \leq \nu(Z_\alpha, \lambda_\alpha))$.

(2) each stratum $S_\alpha = i_\alpha(Y_\alpha^{s.s.}/P_\alpha)$

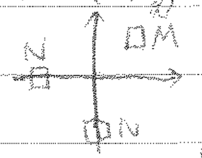
$$X/G = X^{s.s.}(\mathcal{L})/G \cup S_1 \cup \cdots \cup S_n$$

where each S_i parametrizes an unstable pt x + its minimizing test datum (x, λ)

(3) using inner product on cochar, one can define a new elt $\mathcal{L}_\alpha \in NS_{L_{\lambda_\alpha}}(Z_\alpha)_{\mathbb{Q}}$, so that $Z_\alpha^{s.s.}$ is the semistable in $Z_\alpha / L_{\lambda_\alpha}$ w.r.t $\mathcal{L}_\alpha$.

~~Final Picture~~

E.g. $G = GL_2 \quad X = \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^N) \oplus (\det)^{\oplus M}$



$\mathcal{L} = \mathcal{O}(\det)$

minimize $\langle \frac{-\lambda}{|\lambda|}, \chi \rangle$

$\lambda_0 = \begin{bmatrix} t & \\ & t \end{bmatrix} \quad Z_0 = \text{origin} \quad Y_0 = \{(0,0,*)\}$.

$\lambda_1 = \begin{bmatrix} 1 & \\ & t \end{bmatrix} \quad Z_1 = \{(*,0,0)\} \quad Z_1^{s.s.} = \{(\square,0,0) \mid \square \neq 0\}$.

$Y_1^{s.s.} = \{(\square, 0, *)\}$

$\lambda_2 = \begin{bmatrix} t^{-1} & \\ & t^{-1} \end{bmatrix} \quad Z_2 = \cancel{\{(*,*,*\}} \text{ origin}$.

$Y_2 = \{(*,*,*)\} \quad \cancel{Z_2^{s.s.} = \emptyset}$

$Y_2^{s.s.}$ = rest.

Thm Let $\mathcal{X}$ be a locally finite type geom. stack, and $q: \mathcal{X} \to Y$ be f.t. GMS map. Let $l \in NS(\mathcal{X})_{\mathbb{Q}}$ and $b \in H^4(\mathcal{X}, \mathbb{Q})$ be <u>positive definite</u> ($\forall [\cdot/\mathbb{G}_m] \xrightarrow{f} \mathcal{X} \quad f^* b \in \mathbb{Q}_{\geq 0}$ w/ strict ineq. when f nontrivial). Then $\exists$ a stratification determined by HM criterion. $\mathcal{X} = \mathcal{X}^{s.s.} \cup S_1 \cup \dots \cup S_n$ along w/

$S_\alpha \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Z_\alpha^{ss} \quad \pi \circ \sigma \cong id$, s.t.

① $\mathcal{X}^{s.s.}$ is open and $\overline{S_\alpha} \subseteq \bigcup_{\beta \geq \alpha} S_\beta$

② S_α parametrizes families of maps $f: \mathbb{A}^1/\mathbb{G}_m \to \mathcal{X}$ s.t.

$$\nu(f) := -\frac{f^* l}{\sqrt{f^* b}} = \inf_{\substack{f': \mathbb{A}^1/\mathbb{G}_m \to \mathcal{X} \\ f'(1) \cong f(1)}} \nu(f')$$

③ $\mathcal{X}^{s.s.}$ and $Z_\alpha^{s.s.}$ admit good moduli spaces which are proj. over Y.

Furthermore, the stratification is stable under arbitrary base change along $T \to Y$.

pf idea. use Luna slice thm, étale locally $U \xrightarrow{ét} Y$, $\mathcal{X}_U \cong \mathrm{Spec}(R)/G$ for linear reductive G, the main thing to show is that the HM criterion for $\mathrm{Spec}(R)/G$ is local over $\mathrm{Spec}(R^G)$.

Rmk if $\mathcal{X}$ is smooth, then S_α are automatically smooth as well.

Moduli of G-bundles over a curve

Let Σ be a smooth proper curve of genus g.

$\mathcal{M}_G :=$ cat. fibered in gpoids over Sch/k whose fiber over T is the gpoid {principal G-bundles on $T \times \Sigma$}

Prop $\mathcal{M}_G$ is an alg. stack.

pf. $G \hookrightarrow GL_n$

(1) observe $\mathcal{M}_{GL_n} \to Coh(\Sigma)$

$E/T\times\Sigma \longmapsto E \times_{GL_n} \mathbb{A}^n / T\times\Sigma$

this map of stacks is a representable open immersion

(2) $\mathcal{M}_G \to \mathcal{M}_{GL_n}$

$E \longmapsto E \times_G GL_n$

Lemma: G bundle on any $X \longleftrightarrow$ GL_n bundle E'/X and a section of $E'/G \to X$.

(inverse given by pulling back of E along section)