(partial) Notes for :

Course by Harris on Trace formula. &

Seminar on Langlands ...

Time: Spring 2017.

Remark: Where am I ?!!!

What am I doing ?!!!

Trace Formula / Langlands Seminar

Set up: H is a locally cpt, unimodular gp. $\Gamma \subseteq H$ is a discrete subgp. $\Gamma\backslash H$ is assumed to be compact.

Let $f \in C_c(H)$. $R(f) \circlearrowright L^2(\Gamma\backslash H)$:

$$(R(f)\varphi)(x) = \int_H f(y)\varphi(xy)\,dy$$
$$= \int_H f(x^{-1}y)\varphi(y)\,dy$$
$$= \int_{\Gamma\backslash H} \Big(\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\Big)\varphi(y)\,dy$$
$$= \int_{\Gamma\backslash H} K(x,y)\varphi(y)\,dy$$

where $K(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)$.

The summation is over $\gamma \in \Gamma \cap x\,\mathrm{supp}(f)\cdot y^{-1}$, hence finite.

Prop. If $R(f)$ is of <u>trace class</u>, then $\mathrm{tr}(R(f)) = \int_{\Gamma\backslash H} K(x,x)\,dx$

pf. Just expand $K(x,y) = \sum_{i,j} a_{ij}\varphi_i(x)\bar\varphi_j(y)$ in $L^2(\Gamma\backslash H \times \Gamma\backslash H)$.

Then $R(f)$ is of <u>trace class</u> means exactly that $\sum_i a_{ii}$ converges absolutely.

And we see immediately that $\int_{\Gamma\backslash H} K(x,x)\,dx = \sum_i a_{ii}$.

· $\{\varphi_i\}$ is a set of orthonormal basis of $L^2(\Gamma\backslash H)$

Now let $\{\Gamma\}$ be a set of representatives of conjugacy classes of Γ.

Let Γ_γ (resp. H_γ) be the centralizer of γ in Γ (resp. H).

Prop.
$$\mathrm{tr}(R(f)) = \int_{\Gamma\backslash H} K(x,x)\,dx = \int_{\Gamma\backslash H}\sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)\,dx$$
$$= \int_{\Gamma\backslash H}\sum_{\gamma\in\{\Gamma\}}\sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\,dx$$

$$= \sum_{\gamma\in\{\Gamma\}}\int_{\Gamma_\gamma\backslash H} f(x^{-1}\gamma x)\,dx = \sum_{\gamma\in\{\Gamma\}}\int_{H_\gamma\backslash H}\int_{\Gamma_\gamma\backslash H_\gamma} f(x^{-1}u^{-1}\gamma u x)\,du\,dx$$
$$= \sum_{\gamma\in\{\Gamma\}}\int_{H_\gamma\backslash H} f(x^{-1}\gamma x)\,dx\cdot \mathrm{vol}(\Gamma_\gamma\backslash H_\gamma)$$

Fact $L^2(\Gamma\backslash H) = \widehat{\bigoplus}_{\substack{\pi \text{ unitary}\\ \text{irrep of } H}} m(\pi,R)\cdot\pi$

Thm (Selberg's Trace Formula)
$$\sum_{\gamma\in\{\Gamma\}} a_\Gamma^H(\gamma)\, f_H(\gamma) = \sum_{\pi\in\{\text{unitary irrep of } H\}} a_\Gamma^H(\pi)\, f_H(\pi),$$ where

$a_\Gamma^H(\gamma) = \mathrm{vol}(\Gamma_\gamma\backslash H_\gamma)$ $\qquad a_\Gamma^H(\pi) = m(\pi, R)$

$f_H(\gamma) = \int_{H_\gamma\backslash H} f(x^{-1}\gamma x)\,dx$ $\qquad f_H(\pi) = \mathrm{tr}(\pi(f)) = \mathrm{tr}\Big(\int_H f(y)\pi(y)\,dy\Big)$

Cor Let $H = \mathbb{R}$, $\Gamma = \mathbb{Z}$. Then $L^2(S^1) = \widehat{\bigoplus}_n 1\cdot\chi_n$, where $\chi_n : \mathbb{R} \longrightarrow \mathbb{C}^\times$, $x \longmapsto e^{2\pi i n x}$

Let $f \in C_c(\mathbb{R})$, then the formula just says
$$\sum_n f(n) = \sum_n \int f(x)e^{2\pi i n x}\,dx = \sum_n \hat f(n).$$

Cor. Let $|H| < +\infty$. Then $a_\Gamma^H(\gamma) = \frac{|H_\gamma|}{|\Gamma_\gamma|}$ $\quad f_H(\gamma) = \frac{|H|}{|H_\gamma|}\cdot \mathrm{tr}\,\pi(\gamma)$.

Let $f = \mathrm{tr}(\pi)$

$a_\Gamma^H(\pi') = m(\pi', \mathrm{Ind}_\Gamma^H 1) = m(1, \mathrm{Res}_\Gamma^H \pi')$

$f_H(\pi) = \mathrm{tr}\Big(\int_H \mathrm{tr}(\pi)(y)\,\pi'(y)\,dy\Big) = \sum_{h\in H} \mathrm{tr}\,\pi(h)\cdot \mathrm{tr}(\pi')(h)$

$= |H|\cdot \delta_{\pi,\bar\pi'}$.

So the formula says $\sum_{\gamma\in\{\Gamma\}} \frac{|H|}{|\Gamma_\gamma|}\mathrm{tr}\,\pi(\gamma) = |H|\cdot m(1, \mathrm{Res}_\Gamma^H \bar\pi)$.

So $\frac{1}{|\Gamma|}\sum_{\gamma\in\{\Gamma\}}\sum_{\gamma'\in[\gamma]} \mathrm{tr}\,\pi(\gamma') = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\mathrm{tr}\,\pi(\gamma) = m(1, \mathrm{Res}_\Gamma^H\bar\pi)$.

Automorphic Representations

Defn. ~~A~~ Let H be a multiset of cplx numbers of cardinality n, then $f \in \mathcal{A}^\circ_H(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$ is a smooth fctn

$$f : GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}) \longrightarrow \mathbb{C} \quad \text{s.t.}$$

① ~~$f(gh) = f$~~ Right translates of f under $GL_n(\hat{\mathbb{Z}}) \times O(n)$ span a finite dim'l vector space

② $\forall z \in \mathfrak{Z}_n$, then $z \cdot f = \chi_H(\gamma_{HC}(z)) f$, where $\mathfrak{Z}_n$ is the center of $\mathcal{U}(\mathfrak{gl}_n(\mathbb{C}))$ and

$$\gamma_{HC} : \mathfrak{Z}_n \xrightarrow{\sim} \mathbb{C}[x_1, \dots, x_n]^{S_n}$$

③ $\forall\, n = n_1 + n_2$, let $N_{n_1, n_2} = \begin{pmatrix} I_{n_1} & * \\ 0 & I_{n_2} \end{pmatrix} \leq GL_n$, then

$$\int_{N_{n_1,n_2}(\mathbb{Q})\backslash N_{n_1,n_2}(\mathbb{A})} f(ug)\, du = 0$$

④ f is bdd on $GL_n(\mathbb{A})$.

Observe: $\mathcal{A}^\circ_H(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$ admits an action of $GL_n(\mathbb{A}_f) \times (O(n), \mathfrak{gl}_n)$ s.t.
① the stabilizer of any f in $GL_n(\mathbb{A}_f)$ is open.
② the action of $GL_n(\mathbb{A}_f)$ commutes w/ $\mathfrak{gl}_n$ & $O(n)$.
③ $k(Xf) = (kXk^{-1})(kf) \quad \forall k \in O(n)$ and $X \in \mathfrak{gl}_n$
④ the vector space spanned by $O(n)$-translates of any f is finite dim'l.
⑤ if $X \in \mathrm{Lie}(O(n)) \subseteq \mathfrak{gl}_n$, then

$$X(f) = \frac{d}{dt}(e^{tX} f)\Big|_{t=0}$$

Fact $\mathcal{A}^\circ_H(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$ is a direct sum of irred. <u>admissible</u> $GL_n(\mathbb{A}_f) \times (\mathfrak{gl}_n, O(n))$-modules each occurring w/ multiplicity 1.

Defn. A $GL_n(\mathbb{A}_f) \times (\mathfrak{gl}_n, O(n))$-mod V is admissible if $\forall$ irred. (finite dim'l) smooth repn W of $GL_n(\hat{\mathbb{Z}}) \times O(n)$, the space $\mathrm{Hom}_{GL_n(\hat{\mathbb{Z}}) \times O(n)}(W, V)$ is finite dim'l.

The irreducible constituents of $\mathcal{A}^\circ_H(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$ are called cuspidal automorphic repns of $GL_n(\mathbb{A})$ w/ infinitesimal character H.

E.g. $n=1$. Define

$$\|\cdot\| : \mathbb{Q}^\times\backslash\mathbb{A}^\times \longrightarrow \mathbb{R}^\times_{>0}$$
$$(h_v) \longmapsto |h_\infty|_\infty \prod_{v \text{ finite}} |h_v|_v$$

Then $\mathcal{A}^\circ_{\{s\}}(\mathbb{Q}^\times\backslash\mathbb{A}^\times) = \mathcal{A}^\circ_{\{0\}}(\mathbb{Q}^\times\backslash\mathbb{A}^\times) \otimes \|\cdot\|^s$.

And $\mathcal{A}^\circ_{\{0\}}(\mathbb{Q}^\times\backslash\mathbb{A}^\times) = C^\infty_c(\underbrace{\mathbb{Q}^\times\backslash\mathbb{A}^\times/\mathbb{R}_{>0}}_{\cong \hat{\mathbb{Z}}})$

$= \bigoplus \psi \cdot \mathbb{C}$ as $GL_1(\mathbb{A}_f) \times (O(1), \mathfrak{gl}_1)$-mod.

as ψ runs over all continuous characters

$$\psi : \hat{\mathbb{Z}}^\times \longrightarrow \mathbb{C}^\times$$

Nb.
- If $\psi : \mathbb{A}^\times/\mathbb{Q}^\times\mathbb{R}^\times_{>0}$ is a character, ~~then~~ and π is an irred. constituent of $\mathcal{A}^\circ_H(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$ then $\pi \otimes (\psi \circ \det)$ is also an irred. constituent of $\mathcal{A}^\circ_H(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$. We may realize it as fctns of the form $f(g) \cdot \psi\det(g)$ where $f \in \pi$.
- π^* is also an irred. const. of $\mathcal{A}^\circ_{-H}(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$ realized as $f({}^tg^{-1})$ for $f \in \pi$.

Defn
- A smooth repn of $GL_n(\mathbb{Q}_p)$ is $GL_n(\mathbb{Q}_p) \circlearrowright GL(V)$ s.t. $\forall\, v \in V$, $\mathrm{Stab}_v \leq GL_n(\mathbb{Q}_p)$ is open.
- We call V admissible if V^U is finite dim'l for every open $U \leq GL_n(\mathbb{Q}_p)$. Equivalently, $\forall$ irred. (smooth) repn W of $GL_n(\mathbb{Z}_p)$, $\dim \mathrm{Hom}_{GL_n(\mathbb{Z}_p)}(W, V) < \infty$.
- We call an irred. smooth V unramified if $V^{GL_n(\mathbb{Z}_p)} \neq (0)$.

Fact
- Every irred. smooth repn of $GL_n(\mathbb{Q}_p)$ is admissible
- $\forall$ unramified irred. smooth V, $\dim V^{GL_n(\mathbb{Z}_p)} = 1$
- The only finite dim'l irred. smooth repn of $GL_n(\mathbb{Q}_p)$ are 1-dim'l and of the form $\psi \circ \det$ for some $\psi: \mathbb{Q}_p^\times \longrightarrow \mathbb{C}^\times$ continuous.

Defn A $(\mathfrak{gl}_n, O(n))$-module is $V/\mathbb{C}$ w/ $\mathfrak{gl}_n \curvearrowright V$ and $O(n) \curvearrowright V$ s.t.
(1) $k(X \cdot v) = (kXk^{-1}) \cdot (k \cdot v)$ $\forall k \in O(n)$ and $X \in \mathfrak{gl}_n$
(2) the vector space spanned by $O(n)$-translates of any $v \in V$ is finite dim'l.
(3) if $X \in \mathrm{Lie}\, O(n) \subseteq \mathfrak{gl}_n$, then $Xv = \frac{d}{dt}(e^{tX} \cdot v)|_{t=0}$
- We call V admissible if $\forall$ irred $O(n)$-mod W we have $\dim \mathrm{Hom}_{O(n)}(W, V) < \infty$.

Fact. & Defn Any irred admissible $GL_n(\mathbb{A}_f) \times (\mathfrak{gl}_n, O(n))$ module is of the form
$$\pi = \bigotimes' \pi_x$$
where π_p is an irred smooth repn of $GL_n(\mathbb{Q}_p)$, and $\tilde{\forall} p$ π_p is unramified, π_∞ is an irred admissible $(\mathfrak{gl}_n, O(n))$ module. Choose $0 \neq w_p \in \pi_p^{GL_n(\mathbb{Z}_p)}$ $\tilde{\forall} p$, then
$$\bigotimes'_x \pi_x = \varinjlim_S \bigotimes_{p \in S} \pi_p \otimes \pi_\infty \otimes \bigotimes_{p \notin S} (w_p) \subseteq \bigotimes_x \pi_x,$$
which up to isom. doesn't depend on the choice of w_p's

Defn (∞) To an irred admissible $(\mathfrak{gl}_n, O(n))$-mod π_∞ we define
- $\psi_{\pi_\infty}: \mathbb{R}^\times \longrightarrow \mathbb{C}^\times$ w/ $\psi_{\pi_\infty}(-1) = \pi(-I_n)$ and $\psi_{\pi_\infty}(t) = e^{\pi(\log(t) I_n)}$
- $\Gamma(\pi, s) = \prod_{H_j = \{a_j\}} \Gamma_{\mathbb{R}}(s + a_j + \delta_j) \prod_{H_j = \{a_j, b_j\}} \Gamma_{\mathbb{R}}(\max(a_j, b_j) + s) \Gamma_{\mathbb{R}}(\max(a_j, b_j) + 1 + s)$
- $\varepsilon(\pi, e^{2\pi i x}) = \prod_{H_j = \{a_j\}} i^{\delta_j} \prod_{H_j = \{a_j, b_j\}} i^{|a_j - b_j|}$

(p) To any irred smooth repn π_p of $GL_n(\mathbb{Q}_p)$ we associate
- $\psi_{\pi_p}: \mathbb{Q}_p^\times \xrightarrow{\Delta} GL_n(\mathbb{Q}_p) \longrightarrow GL_n(\pi_p)$ which factors $\psi_{\pi_p}: \mathbb{Q}_p^\times \longrightarrow \mathbb{C}^\times$. $\boxed{\text{If } \pi_p \text{ is unramified, then } \psi_\pi(\mathbb{Z}_p^\times) = \{1\},\ \varepsilon(\pi_p, \Psi_p) = 1,\ f(\pi_p) = 0}$
- $L(\pi_p, X) \in \mathbb{C}(X)$
- $\varepsilon(\pi_p, \Psi_p) \in \mathbb{C}^\times$, where $\Psi_p: \mathbb{Q}_p \to \mathbb{C}^\times$ a nontrivial cont. char.
- conductor $f(\pi_p) \in \mathbb{Z}$, it's the minimal non-neg integer $f \in \mathbb{Z}_{\geq 0}$ s.t. $\pi^{U_1(p^f)} \neq (0)$, where
$$U_1(p^f) = \left\{ A \in GL_n(\mathbb{Z}_p) \;\middle|\; A \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod p^f \right\}$$

(Global) To an irred admissible $GL_n(\mathbb{A}_f) \times (\mathfrak{gl}_n, O(n))$-mod π we associate:
- $\psi_\pi = \prod_x \psi_{\pi_x}: \mathbb{A}^\times \longrightarrow \mathbb{C}^\times$
- $L(\pi, s) = \prod_p L(\pi_p, p^{-s})$
- $\Lambda(\pi, s) = \Gamma(\pi_\infty, s) \cdot L(\pi, s)$
- $N(\pi) = \prod_p p^{f(\pi_p)} \in \mathbb{Z}_{>0}$.
- $\varepsilon(\pi) = \prod_x \varepsilon(\pi_x, \Psi_x)$ where $\Psi_p(t) = e^{-2\pi i (t \bmod \mathbb{Z}_p)}$
$\Psi_\infty(t) = e^{-2\pi i t}$

Thm (Godement–Jacquet) π is an irred. constituent of $\mathcal{A}_H^\circ(GL_n(\mathbb{Q}) \backslash GL_n(\mathbb{A}))$ w/ $n > 1$. Then $L(\pi, s)$ converges to a holomorphic fctn in some right half plane $\mathrm{Re}\, s > \sigma$ and can be continued to a holomorphic fctn on the whole cplx plane so that $\Lambda(\pi, s)$ is bdd in all vertical strips $\sigma_1 \geq \mathrm{Re}\, s \geq \sigma_2$. Moreover, we have
$$\Lambda(\pi, s) = \varepsilon(\pi) \cdot N(\pi)^{-s} \Lambda(\pi^*, 1-s).$$

Thm Suppose π & π' are 2 cuspidal automorphic repns of $GL_n(\mathbb{A})$ w/ $\pi_p \cong \pi'_p$ $\tilde{\forall} p$. Then $\pi = \pi'$.

Jacquet–Langlands correspondence for GL_2

Local case F is a local field of char. not 2. D is a quaternion alg. $/F$.

$$GL_2(F) \longrightarrow F \times F^\times \qquad D^\times \longrightarrow F \times F^\times$$
$$\gamma \longmapsto (\mathrm{tr}(\gamma), \det(\gamma)) \qquad \gamma' \longmapsto (\mathrm{Tr}(\gamma'), \mathrm{Nm}(\gamma'))$$

Defn we say a regular semisimple $\gamma \in GL_2(F)$ and $\gamma' \in D'$ $\gamma \sim \gamma'$ if they have the same image in $F \times F^\times$.

Nb. This gives a bij between elliptic regular semisimple in GL_2 and regular semisimple in $D^\times$.

Thm (Local JL) Let ω ~~be~~ $: F^\times \longrightarrow \mathbb{C}^\times$ be a smooth character, then $\exists!$ bij.
$\{$irreducible smooth repn of $D^\times$ w/ central char. $\omega\}$
$\longleftrightarrow \{$irred. discrete series repn of $GL_2(F)$ w/ central char. $\omega\}$
s.t. for $\pi \longleftrightarrow \pi'$ and regular s.s. $\gamma \in D^\times$, $\gamma' \in GL_2(F)$ w/ $\gamma \sim \gamma'$,
we have $\Theta_\pi(\gamma) = -\Theta_{\pi'}(\gamma')$. $\pi' \otimes (\chi \circ \mathrm{Nm}) \longleftrightarrow \pi \otimes (\chi \circ \det)$.

E.g. • $F = \mathbb{R}$ $D = \mathbb{H}$ $D^\times = \mathbb{H}^\times = \mathbb{R}^\times \cdot SU(2)$. Let ω $\mathbb{R}^\times \longrightarrow \mathbb{C}^\times$.
$\omega^+ = \omega|_{\mathbb{R}_{>0}}$. $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ s.t. $\omega(-1) = (-1)^\varepsilon$.
$LHS = \{\omega^+ \boxtimes \mathrm{Sym}^n(\mathbb{C}^2)$ for $n \equiv \varepsilon \pmod 2\}$.
$RHS = \{\omega^+ \boxtimes D_n^\pm$ w/ $n \equiv \varepsilon \pmod 2$ and $n \geq 2\}$
The correspondence gives $\omega^+ \boxtimes \mathrm{Sym}^n(\mathbb{C}^2) \longleftrightarrow \omega^+ \boxtimes D^\pm_{n+2}$.
Character relation: $0 \to \mathrm{Sym}^n(\mathbb{C}^2) \longrightarrow \mathrm{Ind}_{B(\mathbb{R})}^{GL_2(\mathbb{R})}(\chi) \longrightarrow D^\pm_{n+2} \longrightarrow 0$ for some character χ of $T(\mathbb{R})$. If $\gamma \sim \gamma'$, they have the same trace on $\mathrm{Sym}^n(\mathbb{C}^2)$, so it suffices to show $\mathrm{Tr}(\gamma, \mathrm{Ind}_{B(\mathbb{R})}^{GL_2(\mathbb{R})}(\chi))$.
It's because γ is elliptic and have no fixed pt on $\mathbb{P}^1(\mathbb{R}) = \frac{GL_2(\mathbb{R})}{B(\mathbb{R})}$.

• F is local non-arch. $(F, \mathcal{O}_F, \varpi_F, k)$.
Discrete series of $GL_2(F)$ are twisted Steinberg and supercuspidal.
trivial repn of $D^\times$ corresponds to Steinberg repn.

Defn. $0 \to \mathbb{C} \longrightarrow \mathrm{Fun}(\mathbb{P}^1(F)) \longrightarrow St \longrightarrow 0$ defines a repn of $GL_2(F)$ on $St = \mathrm{Fun}(\mathbb{P}^1(F))/\mathbb{C}$.

Character relation: it suffices to show $\mathrm{Tr}(\gamma, \mathrm{Fun}(\mathbb{P}^1(F))) = 0$ $\forall$ elliptic elt $\gamma \in GL_2(F)$. It's b/c γ has no fixed pt on $\mathbb{P}^1(F)$.

More generally, 1-dim'l repn $D^\times \longrightarrow \mathbb{C}^\times$ must factor thru $F^\times$ by the reduced norm ω $D^\times \xrightarrow{\mathrm{Nm}} F^\times \xrightarrow{\chi} \mathbb{C}^\times$.
So $\omega \longleftrightarrow St \otimes (\chi \circ \det)$.

~~Thm~~ ~~(Global JL)~~ **Global Case** Let F be a global field of char. $\neq 2$. Let D be a quaternion alg. $/F$ ramified exactly at places S.

Thm (Global JL) $\exists!$ inj. $\{$irred. automorphic repn of $\mathbb{A}_D^\times$ of dim >1 w/ central char. $\omega\}$
$\hookrightarrow \{$irred. cuspidal automorphic repn of $GL_2(\mathbb{A}_F)$ w/ central char. $\omega\}$.
s.t. $\pi \longleftrightarrow \pi'$ iff $\pi_v \cong \pi'_v$ for $v \notin S$ and $\pi_v \longleftrightarrow \pi'_v$ for $v \in S$.
The image consists exactly of those cuspidal π of $GL_2(\mathbb{A}_F)$ w/ π_v in the discrete series $\forall\, v \in S$.
also, $\pi \longleftrightarrow \pi'$ + smooth char. $\chi: F^\times \backslash \mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times \Rightarrow \pi' \otimes (\chi \circ \mathrm{Nm}) \longleftrightarrow \pi \otimes (\chi \circ \det)$

Applied to $F = \mathbb{Q}$. we may get $\dim S_2^{new}(\Gamma_0(N)) = h(D) - 1$
where $h(D)$ is the ~~class~~ ~~no~~ number of maximal orders in D up to left multiplication by $D^\times$:
$$h(D) = \#\left(D^\times \mathbb{A}_\mathbb{Q}^\times \backslash \mathbb{A}_{D,f}^\times / \prod_p \mathcal{O}_{D_p}^\times\right)$$

Weil-Deligne Repn & Compatible Systems (/$\mathbb{Q}$)

Recall:

$$\begin{array}{ccccccccc} 1 & \to & I_p & \to & Gal_{\mathbb{Q}_p} & \to & (\mathrm{Frob}_p)^{\hat{\mathbb{Z}}} & \to & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \to & I_p & \to & W_p & \to & (\mathrm{Frob}_p)^{\mathbb{Z}} & \to & 1 \end{array}$$

put discrete top.

Frob_p = geom. Frob

Defn p prime, a W-D repn is a pair (r, N) consisting of
- $r : W_p \to GL_n(F)$ is a repn w/ open kernel.
- $N \in \mathrm{End}(F^n)$ s.t. N is nilpotent, and $\forall$ lift ϕ of Frob_p in W_p, $r(\phi) N r(\phi)^{-1} = p^{-1} N$.

Defn an ℓ-int'l W-D repn is a W-D repn/$\overline{\mathbb{Q}}_\ell$ s.t. all eigenvalues of Frob_p have ℓ-adic valuation 0. Equivalently, all eigenvalues of all elements have ℓ-adic val. 0.

Rmk
- $\exists$ equiv of cat. {continuous ℓ-adic repn of $W_{\mathbb{Q}_p}$}

$\updownarrow$

{ℓ-adic W-D repn of ~~$\mathbb{Q}$~~ $W_{\mathbb{Q}_p}$}.

~~(ρ, V)~~ b/c given (ρ, V), we know (by Grothendieck) that $\exists$ open subgp ~~$\mathbb{D}$~~ $H \leq I_p$ s.t. $\rho|_H$ is unipotent. Therefore $\rho(x) = \exp(t_\ell(x) \cdot N)$ for a unique N ~~[illegible]~~ nilpotent $x \in H$.
(~~$\Rightarrow$~~ N depends on the choice of $t_\ell : I_p \to \mathbb{Z}_\ell(1) \simeq \mathbb{Z}_\ell$)
then for a chosen lift φ of Frob_p, we ~~D~~ consider (ρ^φ, V, N) where $\rho^\varphi(\varphi^n x) = \rho(\varphi^n x) \cdot \exp(-t_\ell(x) \cdot N)$.

- A ~~repn of~~ continuous ℓ-adic repn of $W_{\mathbb{Q}_p}$ extends to $G_{\mathbb{Q}_p}$ iff any lift of ~~Frob~~ Frob_p have eigenvalues: ℓ-adic units.

- Therefore we get an equiv of cats.
{continuous ℓ-adic repn of $G_{\mathbb{Q}_p}$} $\leftrightarrow$ {ℓ-int'l ℓ-adic W-D repn of $W_{\mathbb{Q}_p}$}.
this equiv depends on choices of lift of Frob_p & identification of $\mathbb{Z}_\ell(1) \simeq \mathbb{Z}_\ell$. Two different choices differ by a Natural isom.

- $\{\rho_\ell : G_{\mathbb{Q}_p} \to GL_n(\overline{\mathbb{Q}}_\ell)\} \xleftrightarrow{\text{W-D}}$ {ℓ-int'l W-D repn of $G_{\mathbb{Q}_p}/\overline{\mathbb{Q}}_\ell$}

$\updownarrow$ $\overline{\mathbb{Q}}_\ell \simeq \overline{\mathbb{Q}}_{\ell'}$ s.t. Frob. have ℓ-adic val. 0

$\{\rho_{\ell'} : G_{\mathbb{Q}_p} \to GL_n(\overline{\mathbb{Q}}_{\ell'})\} \xleftrightarrow{\text{W-D}}$ {ℓ'-int'l W-D repn of $G_{\mathbb{Q}_p}/\overline{\mathbb{Q}}_{\ell'}$}

Defn (r, N) is called <u>Frobenius semisimple</u> if r is semisimple.

Rmk Given a WD repn, $\exists$ a canonical Frob-s.s. WD repn (r', N) where $r'(\varphi) = r(\varphi)^{ss}$ for a ~~fixed~~ chosen lift φ of Frob_p

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Hodge-Tate repn.

a) let $\chi_p^{cyc} : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times \to \overline{\mathbb{Q}}_p^\times$

b) let $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p} \circlearrowleft G_{\mathbb{Q}_p}$.

c) let $B_{HT} = \mathbb{C}_p[t, t^{-1}] \circlearrowleft G_{\mathbb{Q}_p}$ where $\sigma(t) = \chi_p^{cyc}(\sigma) \cdot t$

Thm (Tate) $(\mathbb{C}_p \cdot t^n)^{G_{\mathbb{Q}_p}} = \begin{cases} \mathbb{Q}_p & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$

Prop. It follows that if $\rho : G_{\mathbb{Q}_p} \to GL_n(\overline{\mathbb{Q}}_p)$, then
$$\dim_{\overline{\mathbb{Q}}_p} (\rho \otimes_{\mathbb{Q}_p} B_{HT})^{G_{\mathbb{Q}_p}} \leq \dim_{\overline{\mathbb{Q}}_p} \rho$$

Defn If equality holds in the above, ρ is called Hodge-Tate

Defn. If $\rho : G_{\mathbb{Q}_p} \to GL_n(\overline{\mathbb{Q}}_p)$ is <u>de-Rham</u> if
$$\dim_{\overline{\mathbb{Q}}_p} (\rho \otimes_{\mathbb{Q}_p} B_{dR})^{G_{\mathbb{Q}_p}} = \dim_{\overline{\mathbb{Q}}_p} \rho$$

Rmk (1) for $n = 1$, $dR \Leftrightarrow HT$
in general, $dR \Rightarrow HT$
(2) for $n = 1$, a p-adic repn ρ is $HT \Leftrightarrow \rho = \eta \cdot (\chi_p^{cyc})^n$ where $\eta : G_{\mathbb{Q}_p} \to \overline{\mathbb{Q}}_p^\times$ is a char. w/ open kernel.

If ρ is HT, then define $HT(\rho) = \{h \mid \dim_{\overline{\mathbb{Q}}_p}(\rho \otimes_{\overline{\mathbb{Q}}_p} t^h \mathbb{C}_p)^{G_{\mathbb{Q}_p}} > 0\}$ as a multiset of $\dim(\rho)$ integers.

E.g. $HT(\chi_p^{cyc}) = \{-1\}$.

Defn A <u>compatible system</u> of λ-adic repn is a collection $\{\rho_{\lambda,\iota} : G_{\mathbb{Q}} \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)\}$ for every λ and $\iota : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ s.t.

(CS1) $\forall (\lambda, \iota)$ $\rho_{\lambda,\iota}$ is de Rham at λ and $HT(\rho_{\lambda,\iota})$ are independent of λ, ι. Call $HT(\rho) = HT(\rho_{\lambda,\iota})$.

(CS2) $\exists$ finite set of primes S s.t. $\forall \lambda, \iota$ $\rho_{\lambda,\iota}$ unramified outside $S \cup \{\lambda\}$.

(CS3) $\forall$ primes p $\exists$ a W-D repn$/\overline{\mathbb{Q}}$ called WD_p w/

$$WD(\rho_{\lambda,\iota}|_{G_p})^{s.s.} \cong (\iota \circ WD_p)$$

E.g.1. Let $\rho : G_{\mathbb{Q}} \longrightarrow GL_n(\overline{\mathbb{Q}})$ be an Artin repn and let

$$\rho_{\lambda,\iota} = \rho \otimes_{\overline{\mathbb{Q}},\iota} \overline{\mathbb{Q}}_\ell .$$

$$HT(\rho) = \{\underbrace{0, \ldots, 0}_{n \text{ of them}}\} \quad \text{and} \quad WD_p = (\rho|_{W_p}, N=0)$$

E.g.2. $\rho_{\lambda,\iota} = \chi_\ell^{cyc}$. $HT(\rho) = \{-1\}$ $WD_p = (r_p, N=0)$

where $r_p : W_p \longrightarrow \mathrm{Frob}_p^{\mathbb{Z}} \longrightarrow \overline{\mathbb{Q}}^\times$

$\mathrm{Frob}_p \longmapsto p^{-1}$.

E.g.3 Let $X/\mathbb{Q}$ - sm. proj. vty. then $H^i_{et}(X, \overline{\mathbb{Q}}_\ell(j))$ form a compatible system.

Thm (Alg. Langlands for $GL_1/\mathbb{Q}$) $\exists$ natural bijection between

$\{$compatible system of 1-dim'l ℓ-adic repn$/\mathbb{Q}\} \longleftrightarrow$

$\{$alg. Hecke char's on $\mathbb{Q}^\times \backslash \mathbb{A}^\times\}$

Recall $\mathbb{Q}^\times \backslash \mathbb{A}^\times \cong \prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0}$. an alg. Hecke char $\varphi = \eta \cdot \|\cdot\|^n$ for $n \in \mathbb{Z}$ and $\eta : \prod_p \mathbb{Z}_p^\times \longrightarrow \mathbb{C}^\times$ w/ open kernel.

given $\varphi = \eta \cdot \|\cdot\|^n$, we associate

$$\{\rho_{\ell,\iota} = (\iota \circ \eta) \cdot (\chi_\ell^{cyc})^n : G_{\mathbb{Q}} \longrightarrow G_{\mathbb{Q}}^{ab} \cong \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \overline{\mathbb{Q}}_\ell^\times\}$$

Conversely, given $\{\rho_{\ell,\iota} : G_{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}}_\ell^\times\}$; by (CS1) we know $\rho_{\ell,\iota}|_{G_{\mathbb{Q}_\ell}} = \eta_{\ell,\iota} \cdot (\chi_\ell^{cyc})^{HT(\rho)}$.

Now by (CS2), we know $\forall$ prime p, $\exists$ $WD_p : W_{\mathbb{Q}_p} \longrightarrow \overline{\mathbb{Q}}^\times$ s.t. $\rho_{\ell,\iota}|_{I_p} = \iota \circ WD_p|_{I_p}$ (call it $\eta_p = WD_p|_{I_p}$)

and $\rho_{\ell,\iota}(\mathrm{Frob}_p) = \iota \circ WD_p(\mathrm{Frob}_p)$.

Let $\eta = \prod \eta_p \cdot \|\cdot\|^{HT(\rho)}$ be the corresponding Hecke char.

From Modular forms to Automorphic Repn / $GL_2(\mathbb{Q})$

$\chi_N: (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ $\quad$ $\underbrace{\mathbb{A}^\times/\mathbb{Q}^\times \cong \mathbb{R}^\times_{>0} \times \hat{\mathbb{Z}}^\times \longrightarrow \hat{\mathbb{Z}}^\times \longrightarrow (\mathbb{Z}/N)^\times \longrightarrow \mathbb{C}^\times}_{\omega}$

① $\omega = \otimes \omega_v$ where $\omega_v: \mathbb{Q}_v^\times \longrightarrow \mathbb{C}^\times$ is a character on $\mathbb{Q}_v^\times$, $\tilde{\forall} v$, ω_v is unramified.

② correspondence {primitive Dirichlet char} $\longleftrightarrow$ {Hecke char of finite order}

$GL_2(\mathbb{R})^{>0}$ $\quad$ $GL_2(\mathbb{Z}) \geq SL_2(\mathbb{Z}) \geq \Gamma_0(N) = \left\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \middle| c \equiv 0 \pmod N\right\}$.

$f \in S_k(\Gamma_0(N), \chi_N, K)$ iff ① $f(\gamma \cdot z) = (cz+d)^k f(z) \cdot \chi_N(d)$ $\forall \gamma \in \Gamma_0(N)$

② f is holomorphic and vanish at all cusps.

Fact $\quad f_p = q + \sum_{n\geq 2} a_n q^n$ a Hecke eigenform, $q = e^{2\pi i z}$ then

$T_p f_p = \lambda_p f_p$ and $\lambda_p = a_p$.

$G = GL_2$ $\quad$ $G(\mathbb{A}) = GL_2(\mathbb{R}) \times \prod_p' GL_2(\mathbb{Q}_p)$

Defn $\quad j(g, \tau) = \det(g)^{-1/2}(c\tau + d)$ $\quad g \in GL_2(\mathbb{R})$

$\forall f \in S_k(\Gamma_0(N), K, \chi_N)$ $\quad \varphi_f(g_\infty) = f(g_\infty \cdot i) \cdot j(g_\infty, i)^{-k}$

for $g_\infty \in GL_2(\mathbb{R})^{>0}$

Prop
- Stabilizer of i is $SO(2, \mathbb{R})$ $\left\{\begin{pmatrix}a&0\\0&a\end{pmatrix}\right\}$
- $\varphi_f(\gamma g_\infty) = \varphi_f(g_\infty) \cdot \chi_N(d)$.
- $\varphi_f(g_\infty K_\theta) = \varphi_f(g_\infty) \cdot e^{i\theta k}$ $\quad K_\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \in SO(2, \mathbb{R})$

$\forall z \in \mathfrak{g} = \mathrm{Lie}\, GL_2(\mathbb{R}) \cong M_2(\mathbb{R})$. $\quad z \cdot \varphi_f(w) = \frac{d}{dt} \varphi_f(w \cdot \exp(tz))\Big|_{t=0}$

This action can be extended to $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$.

$Z_\mathfrak{g} = \{\underbrace{\mathrm{Lie}(Z)}_{\text{center of } GL_2}, \underbrace{C}_{\text{Casimir}}\}$

Thm $\quad SL_2(\mathbb{Q}) \cdot SL_2(\mathbb{R})$ is dense in $SL_2(\mathbb{A})$.

Cor. $\quad GL_2(\mathbb{A}) = GL_2(\mathbb{Q})(GL_2(\mathbb{R}) \cdot K')$, where K' is a open cpt subgp of $GL_2(\mathbb{A}_f)$ s.t. $\det(K') = \hat{\mathbb{Z}}^\times$.

$\Gamma' := GL_2(\mathbb{Q}) \cap (GL_2(\mathbb{R}) K') := \Gamma'$.

e.g. $\quad K' = \prod_p \otimes GL_2(\mathbb{Z}_p)$ $\quad \Gamma' = SL_2(\mathbb{Z})$

$K' = K_0(N) = \left\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \in GL_2(\mathbb{A}_f) \middle| c \equiv 0 \pmod N\right\}$ $\quad \Gamma' = \Gamma_0(N)$

In the latter case, define $\varphi_f(g) = \varphi_f(\gamma \cdot g_\infty \cdot k) = \varphi_f(g_\infty) \cdot \chi(k)$

$\chi: K_0(N) \longmapsto (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$

$\begin{pmatrix}a&b\\c&d\end{pmatrix} \longmapsto a$

Prop $\quad S_k(\Gamma_0(N), \chi_N) \xrightarrow{\sim} \mathcal{A}_0(G)(hol, k, N, \chi)$

where RHS is the space of fctns on $GL_2(\mathbb{A})$ s.t.

(1) $\varphi(\gamma g k) = \varphi(g) \cdot \chi(k)$ $\quad \gamma \in GL_2(\mathbb{Q})$ $\quad k \in K_0(N)$.

(2) $\varphi(g_\infty \cdot g_f)$ is smooth in g_∞ for any fixed $g_f \in GL_2(\mathbb{A}_f)$

(3) $\varphi(g K_\theta) = \varphi(g) \cdot e^{i\theta k}$.

(4) $\int_{\mathbb{A}/\mathbb{Q}} \varphi\left(\begin{pmatrix}1&x\\0&1\end{pmatrix} \cdot g\right) dx = 0$ $\quad \forall g \in GL_2(\mathbb{A})$.

Automorphic Repns: $\quad G = GL_2$

$\mathcal{A}(G, \omega)$ automorphic forms is a set of fctns $\{\varphi\}$

ω is a Hecke char. of finite order

① $\varphi(z \gamma g) = \omega(z) \varphi(g)$ $\quad z \in Z$, $\gamma \in GL_2(\mathbb{Q})$.

② $g_\infty \longmapsto \varphi(g_\infty g_f)$ is smooth on G_∞

③ φ is K_∞-finite, where $K_\infty = SO(2, \mathbb{R})$.

④ $\exists$ cpt open $K' \subseteq G(\mathbb{A}_f)$ s.t. φ is inv under K' (on the right)
⑤ φ is $Z(\mathfrak{g})$-finite ($\mathbb{C}$-finite since Z acts by scalar b/c ①)
⑥ φ is slowly increasing
⑦ $\mathcal{A}_0$ if $\int_{\mathbb{A}/\mathbb{Q}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = 0 \quad \forall g \in G(\mathbb{A})$.

⑥: $x \in GL_2(\mathbb{A}) \longrightarrow \mathbb{A}^5 \quad (x, \det(x)^{-1})$

$y \in \mathbb{A}^5, \quad |y|_v = \max\{|y_i|\}$

$|y| = \prod_v |y|_v$.

slowly increasing: $|\varphi(g)| \le C|g|^N, \quad C > 0$.

F non-arch. $\quad G = GL_2(F)$

$\chi = (\chi_1, \chi_2)$. $\quad \chi\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} = \chi_1(a) \cdot \chi_2(b)$.

$$\begin{array}{cc} P \xhookrightarrow{i} G & B(\chi) = \mathrm{Ind}(\chi) = \left\{ f : G \longrightarrow \mathbb{C} \;\middle|\; f(hg) = \delta^{1/2}(h)\,\chi(h)\, f(g) \quad \forall h \in P \right\} \\ \downarrow & \\ T & \delta\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \frac{|a|}{|b|} \end{array}$$

Prop. · $B(\chi_1, \chi_2) \cong B(\chi_2, \chi_1)$ if $\chi_1^{-1}\chi_2 \ne |\cdot|$ or $|\cdot|^{-1}$.
· If $\chi_1^{-1}\chi_2 \ne |\cdot|$ or $|\cdot|^{-1}$, then $B(\chi_1, \chi_2)$ is irreducible.
· If $\chi_1^{-1}\chi_2 = |\cdot|$, then $B(\chi_1, \chi_2)$ has a 1-dim'l subrep'n $\varphi \circ \det$.
where $\varphi(ab) = \frac{|a|^{1/2}}{|b|^{1/2}} \cdot \chi_1(a) \cdot |b| \cdot \chi_1(b)$

$= |ab|^{1/2} \chi_1(ab)$. $\quad$ So $\varphi = |\cdot|^{1/2} \cdot \chi_1$

Quotient out this 1-dim'l, we get an irrep (Steinberg).
· $B(\chi_1, \chi_2)^\vee = B(\chi_1^{-1}, \chi_2^{-1})$.

Prop. If V is spherical irrep of $GL_2(\mathbb{Q}_p)$, $K = GL_2(\mathbb{Z}_p)$, then V^K is 1-dim'l.

Prop. The only spherical irrep are given by $B(\chi_1, \chi_2)$ or Steinberg corresp to unramified χ_1, χ_2.

Supercuspidal repns. GL_2, $p \ne 2$. Given a ~~quad~~ quadratic extn. L/F, and a char. $L^\times \longrightarrow \mathbb{C}^\times$. we get

$BC(L/F, \chi) : L^\times \longleftrightarrow GL_2(F)$.

Analytic Theory of Drinfeld Module

$X/\mathbb{F}_q$. $\infty \in |X|$. $A = H^0(X - \{\infty\}, \mathcal{O}_X)$. $F = \mathrm{Frac}(A)$. F_∞, $\mathbb{C}_\infty$.
Lattice Λ in $\mathbb{C}_\infty$ is a f.g. A-module of rk r, discrete in $\mathbb{C}_\infty$.

Goal: $\{\text{Drinfeld module over } \mathbb{C}_\infty\}/\sim \;\longleftrightarrow\; \{\text{lattice in } \mathbb{C}_\infty\}/\sim$

$\{A \xrightarrow{\varphi} \mathbb{C}_\infty\{\tau\}$ s.t. $\partial\varphi$ is the given $A \to \mathbb{C}_\infty\}$

$\varphi_1 \sim \varphi_2$ iff $\exists \lambda \in \mathbb{C}_\infty^\times$ s.t. $\lambda\varphi_1 = \varphi_2\lambda$

$\Lambda_1 \sim \Lambda_2$ iff $\exists \lambda \in \mathbb{C}_\infty^\times$ s.t. $\Lambda_1 = \lambda\Lambda_2$.

analogue of Weierstraß's $\wp$:
$$\wp_\Lambda(z) = z \cdot \prod_{\lambda \in \Lambda - \{0\}} \left(1 - \frac{z}{\lambda}\right)$$

Prop.
1) $\wp_\Lambda$ converges on $\mathbb{C}_\infty$, simple zero at Λ.
2) $\wp_\Lambda(x+y) = \wp_\Lambda(x) + \wp_\Lambda(y)$.
3) $\wp_\Lambda : \mathbb{C}_\infty/\Lambda \xrightarrow{\cong} \mathbb{C}_\infty$. $\mathbb{A}^1/\Lambda \xrightarrow{\cong} \mathbb{A}^1$.

Prop. $f : \mathbb{A}^1 \to \mathbb{A}^1$ nonconstant
1) f always has zeros $\Rightarrow$ f surj.
2) $\{\text{zeros of } f\} \cap B(0,r)$ always finite
$f = (z - z_0)^m g(z)$ s.t. $g(z_0) \neq 0$.
3) $f = \prod_{z_i \neq 0} \left(1 - \frac{z}{z_i}\right)^{\mathrm{ord}_{z_i} f} \cdot z^{\mathrm{ord}_0 f} \cdot \lambda$.

$$\wp_\Lambda : \begin{array}{ccc} \mathbb{A}^1/\Lambda & \xrightarrow{\wp_\Lambda} & \mathbb{A}^1 \\ \downarrow \cdot a & & \downarrow \varphi_a \\ \mathbb{A}^1/\Lambda & \xrightarrow{\wp_\Lambda} & \mathbb{A}^1 \end{array}$$

claim: φ_a factors thru $\mathbb{C}_\infty\{\tau\} \subset \mathbb{C}_\infty$

$\varphi_H = \prod_{h \in H} (z - h)$ $\quad H \leq \mathbb{C}_\infty$ finite subgp $\quad$ take $H = \wp_\Lambda(a^{-1}\Lambda)$.

choose c s.t. $c\varphi_H = az + \text{h.o.t.}$
then compare zeros of $\wp_\Lambda(ax)$ and $c\varphi_H(\wp_\Lambda(x))$
both are simple, zeros at $a^{-1}\Lambda$.
Hence $\varphi_a = c\varphi_H \in \mathbb{C}_\infty\{\tau\}$

$\wp_{\lambda\Lambda}(z) = \lambda\wp_\Lambda(\lambda^{-1}z) \Rightarrow \varphi_{\lambda\Lambda}(a) = \lambda \circ \varphi_\Lambda(a) \circ \lambda^{-1}$

bijection: if $a \notin \mathbb{F}_q$. $\wp(ax) = \varphi_a(\wp(x))$ actually determines $\wp$.
$\underbrace{\wp(a^{-1}\Lambda)}_{= a^{-1}\Lambda/\Lambda} = \ker(\varphi_a)$. hence $|\Lambda/a\Lambda| = |A/a|^r$.

Lemma: Λ lattice in $\mathbb{C}_\infty$, then $F_\infty \otimes_A \Lambda \to \mathbb{C}_\infty$ is inj.

$M_I^r(\mathbb{C}_\infty) = \{\text{Drinfeld Module of rk } r \text{ w/ } I\text{-level structure over } \mathbb{C}_\infty\}/\sim$
$\downarrow \pi$
P_I^r = rk r proj. A-module w/ level-I structure (a finite set).

fix $(Y, \alpha) \in P_I^r(\mathbb{C}_\infty)$ $\quad Y \xrightarrow{\cong} \Lambda \quad Y \otimes_A F \cong F^r$
gives $F_\infty^r \hookrightarrow \mathbb{C}_\infty$.

$\{\text{injective maps from } F_\infty^r \to \mathbb{C}_\infty \,/\, \mathbb{C}_\infty^\times\} = \{[x_1, \dots, x_r] \mid \vec{x} \text{ don't lie on any } F_\infty\text{-hyperplane}\}$
$= \mathbb{P}^{r-1}(\mathbb{C}_\infty) \setminus F_\infty\text{-hyperplanes}$
$= \Omega^{r-1}(\mathbb{C}_\infty)$.

we have a map $\Omega^{r-1}(\mathbb{C}_\infty) \longrightarrow \pi^{-1}(Y, \alpha)$.

$$\begin{array}{ccccc} Y & \longrightarrow & F_\infty^r & \xrightarrow{\varphi_1} & \mathbb{C}_\infty \\ \downarrow \gamma & & \downarrow \gamma & \nearrow \varphi_2 & \\ Y & \longrightarrow & F_\infty^r & & \end{array}$$

ambiguous up to $\gamma \in \mathrm{Aut}_A(Y, \alpha)$.

So we get $M_I^r(\mathbb{C}_\infty) = \coprod_{\substack{(Y,\alpha)\\ \in \\ P_I^r}} \mathrm{Aut}_A(Y,\alpha)\backslash \Omega^{r-1}(\mathbb{C}_\infty)$.

Another covering: $P_I^r = GL_r(F)\backslash GL_r(\mathbb{A}_F^f)/K_I$ where
$K_I = \ker(GL_r(\hat{A}) \longrightarrow GL_r(A/I))$ $\quad \hat{A} = \prod_{v\in|X|\setminus\{\infty\}} \mathcal{O}_v$

$GL_r(\mathbb{A}_F^f)/K_I \longleftrightarrow \{\mathcal{E}$ v.b. on $\mathrm{Spec}(A)$ w/ $\mathcal{E}|_I \cong \mathcal{O}^r_{A/I}$ & $\mathcal{E}_F \cong F^r\}$

$\Omega^{r-1}(\mathbb{C}_\infty) \longleftrightarrow \{F_\infty^r \hookrightarrow \mathbb{C}_\infty\}/\mathbb{C}_\infty^\times$

$GL_r(F)\backslash\left(\Omega^{r-1}(\mathbb{C}_\infty) \times \left(GL_r(\mathbb{A}_F^f)/K_I\right)\right) \xrightarrow{\cong} M_r^I(\mathbb{C}_\infty)$

$$\begin{tikzcd}
M \arrow[r] & M\otimes_A F \cong F^r \arrow[r] & F_\infty^r \arrow[r, hook] \arrow[d, "\varphi"] & \mathbb{C}_\infty \\
M' \arrow[r] & M'\otimes_A F \cong F^r \arrow[r] \arrow[u, "\varphi"'] & F_\infty^r \arrow[ur] &
\end{tikzcd}$$

Building $I(K)$ of $PGL(r,K)$

Defn: Let V be a finite dim'l vector space $/K$. A norm on V is a map $V \longrightarrow \mathbb{R}^{\geq 0}$ satisfying...

$I(K)$ = set of norms on $V = H^0(\mathbb{P}_K^{r-1}, \mathcal{O}(1))$ modulo scaling.

$\lambda: \Omega^r \longrightarrow I(K)$

$[x_0, \dots, x_r] \longmapsto \left((a_0, \dots, a_n) \longmapsto |\sum a_i x_i|\right)$

$I(K)$ is a simplicial cplx formed by $\Lambda_0 \supsetneq \Lambda_1 \supsetneq \cdots \supsetneq \Lambda_n = \pi\Lambda_0$, dim $r-1$, $(r-1)$-simplices are called chambers.

$\lambda^{-1}(\text{simplex})$ = affinoids of Ω^r.
any 0-simplex determines a lattice, gives an integral model $\mathbb{P}^{r-1}_{\mathcal{O}_K}$.

Rmk: at places where quaternion alg D ramifies, $Sh_{D,\mathfrak{p}}$ can be covered by Ω^2. (That's why Jie Xia called it Mumford curve...!)

Drinfeld modules to shtukas

given $A \xhookrightarrow{\varphi} R\{\tau\}$ a Drinfeld module.

Let M_i = set of degree $\leq i$ elts of $R\{\tau\}$.

$B = \bigoplus_{i=1}^{\infty} H^0(C, \mathcal{O}_C(i\cdot\infty))$ $\quad (C = \mathrm{Proj}(B))$

Let $N_j = \bigoplus_{i=0}^{\infty} M_{ri+j}$ $\quad N_j$ is $B\otimes_{\mathbb{F}_q} R$-module.

R acts on M_i on the left.

B acts by right multiplication $B_j \subseteq A$.

N_j gives $\mathcal{E}_j$ on $C\times_{\mathbb{F}_q} S(\mathrm{Spec}\, R)$. left mult. by τ gives linear map $\varphi^* N_j \longrightarrow N_{j+1}$.

$F_S^* \mathcal{E}_j \longrightarrow \mathcal{E}_{j+1}$, $\mathcal{E}_j \longrightarrow \mathcal{E}_{j+1}$

Thm · $\mathcal{E}$ is locally free of rk r.
· $H^0(C \times_{\mathbb{F}_q} S, \mathcal{E}_j) \xleftarrow{\cong} M_j$.
· $R\{\tau\}$ itself shall be viewed as a vector bundle on $A \otimes_{\mathbb{F}_q} R$ of rk r over A.
· $\{1, \tau, \dots, \tau^{r-1}\}$ is a basis of lattice on formal punctured disc around $\infty \times S$, gives extension of this vector bundle to $\{\infty\} \times \operatorname{Spec} R$.

Reference Elliptic Modules I, II

Main Thm F = fctn field of a curve/$\mathbb{F}_q$.

Defn $L_0(GL_2(F)\backslash GL_2(\mathbb{A}_F), \overline{\mathbb{Q}}_\ell)$ is the set of fctns

$$f: GL_2(F)\backslash GL_2(\mathbb{A}_F) \xrightarrow{cts} \overline{\mathbb{Q}}_\ell, \text{ s.t.}$$

(1) f is invariant under some open cpt subgp of $GL_2(\mathbb{A}_F)$
(2) $GL_2(F_\infty)\cdot f$ generate a finite direct sum of irrep of $GL_2(F_\infty)$
(3) f is cuspidal $\displaystyle\int_{U(F)\backslash U(\mathbb{A}_F)} f(ux)\,du = 0 \quad \forall x \quad U = \begin{pmatrix}1&*\\0&1\end{pmatrix}$

Rmk Given (1) & (2), then (3) $\Leftrightarrow$ f has cpt support mod center.

Prop (J-L) $L_0(GL_2(F)\backslash GL_2(\mathbb{A}_F)) = \bigoplus_{\pi\in\Pi} \pi$ where Π is a set of irrep of $GL_2(\mathbb{A}_F)$
each π occurs w/ multiplicity 1
Each $\pi = \bigotimes'_v \pi_v$, π_v is irreducible admissible repn of $GL_2(F_v)$ w/ $GL_2(\mathcal{O}_v)$-invariant vector $\forall v$.

Defn. (Special repn of $GL_2(F_\infty)$) $GL_2(F_\infty) \circlearrowright C_c^\infty(\mathbb{P}^1(F_\infty), \overline{\mathbb{Q}}_\ell)/\overline{\mathbb{Q}}_\ell =: V_{sp}$.
(special Galois repn) sp_{Gal} $\quad Gal(F_\infty^s/F_\infty) \longrightarrow \mathbb{Z}_\ell(1)\tau \rtimes \hat{\mathbb{Z}}F$.

$$F \longmapsto \begin{pmatrix}1&0\\0&q_\infty^{-1}\end{pmatrix} \qquad \tau \longmapsto \begin{pmatrix}1&1\\0&1\end{pmatrix}$$

Main Thm: Set $\underline{H} = \varinjlim_H H^1_!(M^2_{H,\bar F}, \overline{\mathbb{Q}}_\ell)$. Then as $GL_2(\mathbb{A}_F)\times Gal(F^s/F)$-module, we have

(0) $\underline{H} = \bigoplus_{\substack{\pi\in\Pi \\ \pi_\infty = V_{sp}}} \pi\otimes\sigma(\pi)$ where $\sigma(\pi)$ is a 2-dim'l Gal repn and $\sigma(\pi)|_{D_\infty} = sp_{Gal}$. D_∞ = decomposition gp at ∞.

(1) $\sigma(\pi\otimes\chi) = \sigma(\pi)\otimes\chi$ where we identify $Gal(F^{s,ab}/F) \cong \widehat{GL_1(\mathbb{A}_F)/GL_1(F)}$

(2) $\det(\sigma(\pi)) = \omega_\pi^{-1}(-1)$ where ω_π = character of $Gal(F^s/F)$ associated to $\pi|_{center}$.

(3) If v is a place s.t. π_v is unramified ($\exists\, GL_2(\mathcal{O}_v)$-fixed vector) and $\sigma(\pi)|_{Gal(F_v^s/F_v)}$ is unramified. Then "Hecke poly" equals the char. poly of $Frob_v \circlearrowright \sigma(\pi)$.

Rmk. Unramified $\Rightarrow$ $\pi_v = Ind_B^G(\chi_1,\chi_2)$ and χ_1 and χ_2 are unramified.
$Ind_B^G(\rho) = \{f: G\to \mathbb{C} \mid f(gh^{-1}) = \rho(h)\, f(g)\}$.
Then $(T-\chi_1(rec^{-1}(Frob_v)))(T-\chi_2(rec^{-1}(Frob_v)))$ is the Hecke poly.

for (0) as $GL_2(\mathbb{A}_F)\times Gal(F_\infty^s/F_\infty)$-module:

$$\underline{H} \cong Hom_{GL_2(F_\infty)}(V_{sp}, L_0(GL_2(\mathbb{A}_F))) \otimes sp_{Gal}.$$

for (3) Eichler–Shimura relations:
Y-modular curve of level prime to p. ($Y(N)$, $p\nmid N$) sm. curve/$\mathbb{Z}_p$
$T_p \circlearrowright H^1_!(Y(N), \overline{\mathbb{Q}}_\ell)$. $T_p = \{(E_1,\varphi_1,E_2,\varphi_2) \mid v: E_1\to E_2$, p-isogeny, respecting $\varphi_1, \varphi_2\}$.
$\|$
$Frob_p \circlearrowright H^1_!(Y(N)_{\bar{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$

Thm $T_p = F + V$.

pf. $T_p = \Gamma_{Frob} \cup V$ over $\bar{\mathbb{F}}_q$
$F = \Gamma_{Frob} = \{(E_1,\varphi_1, E_1^{(p)}, \varphi_1^{(p)}) \mid Frob_p: E\longrightarrow E^{(p)}\}$
$V = \qquad\qquad V: E^{(p)} \longrightarrow E\}$.

Geometric Langlands correspondence

Set up

X smooth proj. curve / $k = \mathbb{F}_q$, $F = k(X)$

G = conn. reductive gp / F, split $G = G_0 \times_k F$

further assume $G = GL_n, SL_n, PGL_n, Sp(V), SO(W)$

$G(\mathbb{A}) = \prod_v' G(F_v) \supseteq G(F)$ relative to $\prod_v G(\mathcal{O}_v)$

$N = \sum n_v v$ effective divisor.

$K_N = \{k \in \prod_v G(\mathcal{O}_v) \mid k \equiv 1 \pmod{\mathfrak{m}_v^{n_v}} \text{ all } v\}$.

coefficient ring : usually $A = \begin{cases} \overline{\mathbb{Q}}_\ell \\ \mathbb{C} \end{cases}$, $\ell \neq p$

$$\mathcal{A}(G, K_N, A) = C_c(G(F)\backslash G(\mathbb{A})/K_N, A).$$
$$= \mathcal{A}(G, A)^{K_N}$$

$$\mathcal{A}(G, A) = C_c(G(F)\backslash G(\mathbb{A}), A) \circlearrowleft G(\mathbb{A})$$

$f \in \mathcal{A}(G,A)$, $g \in G(\mathbb{A})$, $g \cdot f(h) = f(hg)$.

$\mathcal{A}(G,A) \supseteq \mathcal{A}_0(G,A)$ cusp forms.

If G not ~~split~~ semisimple...

$\mathcal{A}_0(G,A) \cong \bigoplus_\pi n(\pi)\cdot\pi$, π runs thru admissible irrepn of $G(\mathbb{A})$

Q: determine $n(\pi)$. For $G = GL_n$, $n(\pi) = 1$ or 0.

Fixing N, question becomes determine π w/ $n(\pi) > 0$, s.t. $\pi^{K_N} \neq 0$.

Thm (VL) $A = \begin{cases} \overline{\mathbb{Q}}_\ell \\ \mathbb{C} \end{cases}$ $\mathcal{A}_0(G, K_N, A) = \bigoplus_\sigma \mathcal{A}_{0,\sigma}$ where σ runs thru Langlands parameter.

$\Gamma = \mathrm{Gal}(F_N/F)$, F_N = max. sep extn of F unramified outside supp(N), a Langlands parameter is a homomorphism
$\sigma : \Gamma \longrightarrow \hat{G}(A)$, up to conjugate by $\hat{G}(A)$.

Main pt of VL's thm, decomposition is compatible w/ local Langlands correspondence for unramified places:

$\exists$ $\mathcal{B}_N$ commutative algebra of excursion operators
to any $h: B_N \longrightarrow A$, VL assigns a Langlands parameter
$\sigma_h : \Gamma \longrightarrow \hat{G}(\bar{\mathbb{Q}}_\ell)$ s.t.
$\chi_V(\sigma_h(\mathrm{Frob}_v)) = T(h_{V,v})$.

Satake Isomorphism.

G split reductive gp / field A, char $A=0$.
$T \subseteq G$, max'l torus, G is determined by $(X, \Phi, X^\vee, \Phi^\vee)$
$X = \mathrm{Hom}(T, \mathbb{G}_m)$. $\Phi = \Phi(G,T)$ set of roots $\subseteq X$.
$X^\vee = \mathrm{Hom}(\mathbb{G}_m, T)$, $\Phi^\vee$ = set of coroots $\subseteq X^\vee$.
$\alpha \in \Phi \rightsquigarrow T_\alpha = \ker(\alpha)^\circ \subseteq T$
$H_\alpha = Z_G(T_\alpha)$ $\quad$ Lie $H_\alpha = T \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ $\quad$ $H_\alpha^{\text{dual}} = SL(2)$ / $PGL(2)$.
$\alpha^\vee : \mathbb{G}_m \longrightarrow T \subseteq H_\alpha \subseteq G$ s.t.
$T = \mathrm{Im}(\alpha^\vee) \cdot T_\alpha$
$\langle\ ,\ \rangle : X \times X^\vee \longrightarrow \mathbb{Z}$. $\quad$ $\langle \alpha, \alpha^\vee \rangle = 2$.

Thm The map taking G to $\Phi(G)$ is a bijection
$$\left\{ \begin{array}{c} \text{Isom. classes of split} \\ \text{conn. reductive groups} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Isom. classes of} \\ \text{root datum} \end{array} \right\}.$$

Defn $\hat{G} = \Phi^{-1}(\Phi(G)^\vee)$.

Fact (1) Suppose G semisimple, Let $P(\Phi) \subseteq X \otimes \mathbb{Q}$ generated by highest weight of irreps of G, $Q(\Phi) \subseteq X$, Lattice generated by Φ. Then G is simply connected $\Leftrightarrow$ $P(\Phi) = X$
adjoint $\quad Q(\Phi) = X$.

(2) Moreover $P(\Phi)$ is dual to $Q(\Phi^\vee)$.
$G \supseteq B \supseteq T$ choosing a Borel $\Leftrightarrow$ choosing a set of positive roots $\alpha \in \Phi$ s.t. $\mathfrak{g}_\alpha \subseteq \mathrm{Lie}(B)$.
$\Phi^+(B) \sqcup -\Phi^+(B) = \Phi$.
$\Leftrightarrow$ choosing a basis $\Delta \subseteq \Phi$ of positive roots, s.t. every $\alpha \in \Phi^+$ is a positive linear combination of Δ.

$(G,B,T) \longleftrightarrow (X,\Delta,X^\vee,\Delta^\vee)$.

Defn A pinning of the root datum is a choice of basis $X_\alpha \in \mathfrak{g}_\alpha$ for all $\alpha \in \Delta$.

$$1 \longrightarrow \mathrm{Int}(G) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1.$$

$\left\{ \begin{matrix} g \mapsto hgh^{-1} \\ h \in G \end{matrix} \right\}$ after choosing a pinning we'll have this

$\mathrm{Out}(G) \simeq \mathrm{Aut}(G,B,T,\{X_\alpha\})$.

Any two pinning differ by $\mathrm{Int}(t)$ some $t \in T$.

Defn A Langlands parameter for G is a homomorphism

$$\begin{tikzcd} \mathrm{Gal}(\bar{A}/A) \arrow[r] \arrow[dr, "\mathrm{Id}"'] & {}^L G(\bar{\mathbb{Q}}_\ell) = \hat{G} \rtimes \mathrm{Gal}(A'/A) \arrow[d] \\ & \mathrm{Gal}(A'/A) \end{tikzcd}$$

Defn G/ global function field $F = k(X)$ $\mathcal{A}(G,A) = C_c(G(F)\backslash G(\mathbb{A}), A)$.

An automorphic repn is an irred. $G(\mathbb{A})$-invariant subspace of $\mathcal{A}(G,A)$. In particular π is an irred. repn of $G(\mathbb{A})$ that is smooth: $\forall\, v \in \pi$, $\exists \underset{\text{open subgp}}{U \subseteq} G(\mathbb{A})$ s.t. $v \in \pi^U$.

Thm Any irred. repn of $G(\mathbb{A})$ is isomorphic to restricted tensor product $\bigotimes_v' \pi_v$, where (a) π_v is an irrep smooth of $G_v = G(F_v)$

(b) for all but finitely many v, $\pi_v^{G(\mathcal{O}_v)} \neq 0$ (π_v is spherical)

(c) $\bigotimes_v' \pi_v = \varinjlim_{S \subseteq X} \bigotimes_{v \in S} \pi_v \otimes \bigotimes_{v \notin S} \underset{\dim = 1}{\pi_v^{G(\mathcal{O}_v)}}$ $\longleftarrow$ vectors unramified outside of S.

Starting point: classify π_v w/ $\pi_v^{G(\mathcal{O}_v)} \neq 0$.

Theory of maximal compact subgp of $G(F_v)$.

Defn $\mathcal{H}(G_v,K_v) = C_c(K_v \backslash G_v / K_v, A)$, e.g. $G = GL_n$, $K_v = G(\mathcal{O}_v)$.

$G_v = \coprod_{t \in A^{++}} K_v t K_v$, $(\varpi_v) = m_v$, $A^{++} = \left\{ \begin{pmatrix} \varpi_v^{a_1} & & 0 \\ & \ddots & \\ 0 & & \varpi_v^{a_n} \end{pmatrix} \right\}$ $a_1 \geq a_2 \geq \cdots \geq a_n$

$\mathcal{H}(G_v,K_v) = \bigoplus_{t \in A^{++}} A t$.

If π is spherical, $\mathcal{H}$ acts by convolution on π^{K_v}, hence π defines a character $\lambda_\pi : \mathcal{H} \longrightarrow A$.

Let $B \subseteq G_v$ integral Borel subgp, then $G_v = B \cdot K_v$ (Iwasawa decomp.)

Thm Satake isomorphism identifies $\mathcal{H}$ w/ $A[X^\vee]^W$.

$\pi \subseteq \mathcal{A}_0(G, \bar{\mathbb{Q}}_\ell) \xrightarrow{VL} \sigma_\ell : \Gamma \longrightarrow \hat{G}(\bar{\mathbb{Q}}_\ell)$

$\pi^{K_N} \neq 0$ unramified outside N

$X \setminus N = X \setminus \mathrm{supp}(N)$, Fix a non-neg. integer $n \geq 0$ $\hat{G}^n$

$\hat{G}^n \longrightarrow \underset{\text{irred.}}{\mathrm{Aut}(W)}$ $W = \bigotimes_{i \geq 1}^{n} W_i$

To this data, VL associates a moduli stack $\mathrm{Cht}^n_{N,W}$, which is a DM stack.

Definition $\forall$ S, a test scheme $/k$, we get the S-valued pt

- An n-tuple $x = (x_i, i = 1, \cdots, n)$ of S-valued pt of $X \setminus N$
- A G torsor g over $X \times S$.
- An isom: $\varphi : g \xrightarrow{\sim} {}^{\tau}g$ away from $\bigcup \Gamma_{x_i}$.
- relative position of φ at each x_i is bdd by the dominant weight of W_i.

The paws define a morphism $p : \mathrm{Cht}^n_{N,W} \longrightarrow (X \setminus N)^n$.
Fix a bound on the H-N polynomial μ,
$\mathrm{Cht}^{n, \leq \mu}_{N,W}$ is of finite type.

Defn $R^\bullet p_! IC_{\mathrm{Cht}^{n,\leq\mu}_{N,W}}$, a sheaf $\mathcal{H}^{\leq\mu}_{N,n,W}$ over $(X \setminus N)^n$, get an action of $\pi_1((X \setminus N)^n)$ on this. Via Drinfeld's lemma, one actually gets an action of $(\pi_1(X \setminus N))^n$

Thm. (1) For any n, if $W = 1$, then $\mathrm{Cht}^n_{N,1}$ is the constant discrete stack $G(F) \backslash G(\mathbb{A}) / K_N$ over $(X \setminus N)^n$

$$\varinjlim_{\mu} \mathcal{H}^{\leq\mu}_{N,n,1} = C_c(G(F) \backslash G(\mathbb{A}) / K_N, \bar{\mathbb{Q}}_\ell).$$

(2) the map $W \longmapsto \mathcal{H}^{\leq\mu}_{N,n,W}$ extends to an additive function on $\mathrm{Repn}(\hat{G}^n)$

(3) the stalk of $\varinjlim_{\mu} \mathcal{H}^{\leq\mu}_{N,n,W}$ at a (good) geometric general point, contains a subspace $H_{n,W}$ of Hecke finite cohomology, which equals $\mathcal{A}_0$ if $W = 1$.

(4) Each $H_{n,W}$ carries an (monodromy) action of $\mathrm{Gal}(F^s/F)^n$ unramified outside N.

(5) Given any morphism $\zeta : [m] \longrightarrow [n]$, there is a natural projection $[\zeta] : \hat{G}^n \longrightarrow \hat{G}^m$, thus $W \in \mathrm{repn}(\hat{G}^m)$ defines $W^\zeta \in \mathrm{Repn}(\hat{G}^n)$, $\exists$ canonical isomorphism $\chi_\zeta : H_{m,W} \simeq H_{n,W^\zeta}$ equivariant for $\mathrm{Gal}(F^s/F)^n$, $\chi_{\zeta_1 \circ \zeta_2} = \chi_{\zeta_1} \circ \chi_{\zeta_2}$.

(6) $\exists$ canonical action of the Hecke alg. $\mathcal{H}(G(\mathbb{A}), K_N)$ on each $H_{n,W}$, all morphisms commute with this.

extension ~~Defn~~ Drinfeld $G = GL_n$ 2 paws
L. Lafforgue

compute $H_{2, \mathrm{Std}, \mathrm{St}^*}$ as a Galois repn using Lefschetz + Arthur–Selberg trace formula.

Defn $S_{n,f,\gamma} \in \mathcal{B}_N$, n as before, $\gamma = (\gamma_1, \cdots, \gamma_n) \in \Gamma^n$.
$f \in \mathcal{O}(\hat{G}^n /\!/ \hat{G}) = \mathcal{O}[\hat{G}^n]^{\hat{G}}$, $\hat{G}$ acts by conjugation,
f can be represented by a tuple $W \in \mathrm{Repn}(\hat{G}^n)$, $x \in W^{\hat{G}}$,
$\xi \in (W^\vee)^{\hat{G}}$: $f(g_1, \cdots, g_n) = \xi((g_1, \cdots, g_n) \cdot x)$.
$x : 1 \longrightarrow W$ $\quad \xi : W \longrightarrow 1$. $\quad \hat{G}$ invariant.
$\delta : \hat{G} \longrightarrow \hat{G}^n$, get $x : 1 \longrightarrow W^\delta$ $\quad \xi : W^\delta \longrightarrow 1$ as $\hat{G}$-repn.

$S_{n,f,\gamma} \in \mathrm{End}(\mathcal{A}_0(G, K_N, \bar{\mathbb{Q}}_\ell)) = \mathrm{End}(H_{0,\underline{1}}) = \mathrm{End}(H_{1,\underline{1}})$.

$$\begin{tikzcd}
H_{1,\underline{1}} \arrow[r, "{[x]}"] \arrow[drr, "S_{n,f,\gamma}"'] & H_{1,W^\delta} \arrow[r, "\chi_\delta", "\sim"'] & H_{n,W} \arrow[r, "{(\gamma_1,\cdots,\gamma_n)}"] & H_{n,W} \arrow[d, "\chi_\delta^{-1}"] \\
 & & H_{1,\underline{1}} & H_{1,W^\delta} \arrow[l, "{[\xi]}"']
\end{tikzcd}$$

Thm. $S_{n,f,\gamma}$ form a commutative $\bar{\mathbb{Q}}_\ell$ subalg. $\mathcal{B} \subseteq \mathrm{End}(\mathcal{A}_0(G, K_N, \bar{\mathbb{Q}}_\ell))$
satisfying (1) for n, γ fixed, $f \longmapsto S_{n,f,\gamma}$ is an alg. hom.
(2) $S_{n,f,\gamma}$ satisfy natural relations with respect to
$[\zeta] : \hat{G}^n \longrightarrow \hat{G}^m$ $\quad [\zeta]_\Gamma : \Gamma^n \longrightarrow \Gamma^m$.
(3) — ~~ multiplication of 2 elements in $\hat{G}^n$, Γ^n.
(4) fix n, f. $\Gamma^n \longrightarrow \mathrm{End}(\mathcal{A}_0(G, K_N, \bar{\mathbb{Q}}_\ell))$ is continuous in the ℓ-adic topology.
(5) The unramified Hecke alg. $T_N \subseteq \mathcal{B}_N$.
$V \in \mathrm{Repn}(\hat{G})$ irred.
$\Downarrow$
$h_{V,v} \in T_N$ $\qquad S_{2, f_V, (\mathrm{Frob}_v, 1)} = h_{V,v}$
$f_V(g_1, g_2) = \mathrm{Tr}_V(g_1 g_2^{-1})$.

Let $\nu : \mathcal{B}_N \longrightarrow \bar{\mathbb{Q}}_\ell$ a character, then the data $(n, f, \gamma) \longmapsto \nu(S_{n,f,\gamma})$ is a $\bar{\mathbb{Q}}_\ell$ valued $\hat{G}$-pseudocharacter of Γ.

Thm If $r : \Gamma \longrightarrow \hat{G}(\bar{\mathbb{Q}}_\ell)$ is a homomorphism, for $\mathrm{tr}(r) = (\Theta_n)_{n \geq 1}$.
$\Theta_n(f)(\gamma_1, \cdots, \gamma_n) = f(r(\gamma_1), \cdots, r(\gamma_n))$, $f \in \mathcal{O}[\hat{G}^n]^{\hat{G}}$.
And every pseudocharacter arise in this way.

pseudorepns and pseudochar. for $G = GL_n$

Defn Let R be a top. ring, G a top. gp w/ unit e. An <u>R-valued pseudorepn</u> of G of dim $d \in \mathbb{N}$ is a cont. fctn $T: G \to R$ that satisfies ① $T(e) = d$ and $d!$ invertible in R.

② $g_1, g_2 \in G$, $T(g_1 g_2) = T(g_2 g_1)$

$(\Lambda^{d+1} V = 0)$ ③ $d \geq 0$ is the smallest integer s.t.:

Let S_{d+1} be the symm gp, $\mathrm{sgn}: S_{d+1} \to \{\pm 1\}$, $\forall (g_1, \cdots, g_{d+1}) \in G^{d+1}$,

$$\sum_{\sigma \in S_{d+1}} \mathrm{sgn}(\sigma) T_\sigma(g_1, \cdots, g_{d+1}) = 0.$$

where if $\sigma_1, \cdots, \sigma_s$ is the cycle decomposition

$$\sigma_j = (i_1^{(j)}, \cdots, i_{r_j}^{(j)}) \text{ of length } r_j, \ \sum r_j = d+1$$

then $T_\sigma(g_1, \cdots, g_{d+1}) = \prod_{j=1}^{s} T(g_{i_1}^{(j)} \cdots g_{i_{r_j}}^{(j)}) = 0.$

Thm (a) suppose $\rho: G \to GL(d, R)$ is a continuous repn, then $\mathrm{tr}(\rho)$ is a d-dim'l pseudorepn.

(b) conversely, if R is an alg. closed field of char. 0 or $p > d$, then any d-dim'l pseudorepn valued in R is the trace of a semisimple repn dim d, unique up to equivalence.

Rmk. • Brauer–No... thm: a semisimple repn is determined by its char.

What's the point? Suppose you have a function $T: G \to \mathcal{O}$, $\mathcal{O}$ a p-adic integer ring, m maximal ideal, $\forall r$, $T_r: G \to \mathcal{O}/m^r$ is the trace of an $\mathcal{O}/m^r$-valued repn $T_r(\rho_r) = \mathrm{Tr}(\rho_{r+1}) \pmod{m^r}$

Then $T = \varprojlim T_r$ is a d-dim'l pseudorepn.

hence $\exists\ \rho: G \to GL(d, \overline{\mathrm{Frac}(\mathcal{O})})$ (used to construct Galois repn)

VL's construction yields, $\forall\ \nu: \mathcal{B}_N \to \mathcal{O}_\ell$, a $\overline{\mathbb{Q}}_\ell$-valued $\hat{G}$-pseudochar. of $\Gamma = \mathrm{Gal}(F^N/F)$, more precisely, $\forall$ n-tuple $(\gamma_1, \cdots, \gamma_n) \in \Gamma^n$, we get a geom. point $\xi_n(\gamma_1, \cdots, \gamma_n) \in \hat{G}^n /\!/ \hat{G}$.

The geom. points of $\hat{G}^n /\!/ \hat{G}$ over an alg. closed field $\longleftrightarrow$ closed orbits / action, i.e., $\xi_n(\gamma_1, \cdots, \gamma_n): \mathcal{O}[\hat{G}^n]^{\hat{G}} \to \overline{\mathbb{Q}}_\ell$.

More precisely, given any I index set, $f \in \mathcal{O}[\hat{G}^n]^{\hat{G}}$, $\gamma = (\gamma_1, \cdots, \gamma_n)$, get $\nu(S_{I, f, (\gamma_1, \cdots, \gamma_n)}) \in \overline{\mathbb{Q}}_\ell$, called it $\nu(I, f, (\gamma_1, \cdots, \gamma_n))$.

Take $I = \{0, \cdots, n\}$ $\quad \hat{G}^n /\!/ \hat{G} \xrightarrow{\sim} \hat{G} \backslash \hat{G}^I / \hat{G}$

$$(g_1, \cdots, g_n) \mapsto (1, g_1, \cdots, g_n)$$

$\mathcal{O}[\hat{G}^n]^{\hat{G}} \simeq \mathcal{O}[\hat{G} \backslash \hat{G}^I / \hat{G}]$

$\Theta_n^\nu: \mathcal{O}[\hat{G}^n]^{\hat{G}} \to C(\Gamma^n, \overline{\mathbb{Q}}_\ell)$ $\quad f \mapsto \nu(I, f, (\gamma))$

satisfying: ① $\forall n$, Θ_n^ν is a continuous algebra homomorphism.

② $\zeta: \{1, \cdots, m\} \to \{1, \cdots, n\}$ a set map. For $f \in \mathcal{O}[\hat{G}^m]^{\hat{G}}$, let $f^\zeta \in \mathcal{O}[\hat{G}^n]^{\hat{G}}$ defined by $f^\zeta(g_1, \cdots, g_n) = f(g_{\zeta(1)}, \cdots, g_{\zeta(m)})$.

Then $\Theta_n^\nu(f^\zeta, (\gamma_1, \cdots, \gamma_n)) = \Theta_m^\nu(f, (\gamma_{\zeta(1)}, \cdots, \gamma_{\zeta(m)}))$.

③ For $f \in \mathcal{O}[\hat{G}^n]^{\hat{G}}$, define $f^{+1} \in \mathcal{O}[\hat{G}^{n+1} /\!/ \hat{G}]$ by

$$f^{+1}(g_1, \cdots, g_{n+1}) = f(g_1, \cdots, g_n \cdot g_{n+1})$$

Then $\Theta_{n+1}^\nu(f^{+1})(\gamma_1, \cdots, \gamma_{n+1}) = \Theta_n^\nu(f)(\gamma_1, \cdots, \gamma_n)$.

VL: The datum $\{\Theta_n^\nu\}$ are $\hat{G}$-pseudocharacters.

cuspidal form

$K_N \subseteq \prod_v G(\mathcal{O}_v)$ coefficients char. 0.

$$\begin{array}{ccc} \mathcal{A}(G, K_N) & = & C_c(G(F)\backslash G(\mathbb{A})/K_N). \\ \cup\!| & & \cup\!| \\ \mathcal{A}_0(G, K_N) & \subseteq & C_0(G(F)\backslash G(\mathbb{A})/K_N) \end{array}$$

Thm (Harder) all cusp forms / function field are compactly supported.

Thm (VL) any ~~compt~~ cptly supported functions that are Hecke finite are cusp forms.

$T \in \mathcal{H}(G, K_N) = C_c(K_N\backslash G(\mathbb{A})/K_N) = \bigotimes_v{}' C_c(K_N\backslash G_v/K_N)$ $K_N = \prod K$

$T(f)(g) = \int_{G(\mathbb{A})} f(gh)\, T(h)\, dh.$

I don't understand anything, algebraic groups are important!!!