

ALTERATIONS AND SEMISTABLE REDUCTION
(MCM–YMSC p -ADIC GEOMETRY LEARNING SEMINAR, FALL 2024)

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Purpose: The main goal of this seminar is to study alterations and semistable reduction theorems of varieties.

Given an algebraic variety X over a field K , does there exist a birational proper map $f: X' \rightarrow X$ such that X' is regular?

Hironaka proved that this is true if K has characteristic 0. In this case he showed that f can be arranged such that f is an isomorphism over the regular locus $X_{\text{reg}} \subset X$ and $f^{-1}(X \setminus X_{\text{reg}})$ is a simple normal crossing divisor. This is a deep theorem, and the positive characteristic case is still wide open. One can consider variants of the statement:

- (1) (alteration) only require f to be generically finite, proper, and surjective;
- (2) (semistable reduction) work on X/K smooth proper with K the fraction field of a DVR R and ask whether $X \otimes_K K'$ admits a semistable model for some finite K'/K .

Correspondingly, the seminar studies (1) de Jong's theorem on alterations [dJ96] and Hartl's local alteration in rigid geometry, and (2) the semistable reduction theorem in characteristic 0 (i.e. the residue field of R has characteristic 0) by Kempf–Knudsen–Mumford–Saint-Donat [KKMSD73].

Time: 2:30-4:00 pm on Mondays

Location: MCM 110

Website: <http://shizhang.li/seminars/alterationsandsemistablereduction.html>; <https://ymsc.tsinghua.edu.cn/info/1053/3152.htm>

Mailing List: We make the seminar announcements via the mailing list. To join the mailing list, please contact Koji.

Schedule: The following is an outline and suggestion for each talk. Sometimes, too many topics are assigned to one talk. Please reorganize the materials to give a 90-minute-long *comprehensible* talk, rather than copying the references. Junior speakers are encouraged to talk with us during the lecture preparation.

Lecture 1. *Overview.* (9/9, Koji)
Give an overview of the seminar.

No meeting on 9/16.

The next three lectures discuss [dJ96]. Note that many arguments therein will be used in Lecture 5. One can also consult Berthelot's Bourbaki talk [Ber97] or B. Conrad's course <https://virtualmath1.stanford.edu/~conrad/249BW17Page/>.

Lecture 2. *Semistable curves and normal crossing divisors.* (9/23, Shizhang)
Discuss [dJ96, §3]. Review materials in §2 if necessary and explain at least one of the local blow-up computations in detail. End the talk with the discussion in §2.24.

Date: August 28, 2024.

No meeting on 9/30 and 10/7.

Lecture 3. *Alterations of varieties.* (10/14,)

The main goal is to prove [dJ96, Thm. 4.1]. Don't spend too much time on the reductions in §4.5-4.10. Explain the techniques in §2.11, and discuss the main ideas and outline of the proof of Theorem 4.1.

Lecture 4. *Semistable alterations.* (10/21,)

The main goal is to prove [dJ96, Thm. 6.5]. First discuss Lemmas 5.2 & 5.6 and Theorem 5.8, and then move to §6. As in Lecture 3, discuss the main ideas and outline of the proof of Theorem 6.5. Focus more on the arguments that are not covered in Lecture 3.

Lecture 5. *Semistable models for rigid-analytic spaces.* (10/28,)

The main goal is to prove [Har03, Thm. 1.4, Cor. 1.5]. Focus more on the arguments that are different from [dJ96]: e.g., Lemmas 1.7 & 1.8, the reduced fiber theorem, Lütkebohmert's compactification theorem. Organize the contents carefully so that the talk is comprehensible.

The next three lectures serve as an introduction to toric varieties. We follow [KKMSD73, Chap.1], in which they are called torus embeddings. One can also consult [Ful93], for example. The speakers are expected, at least, to mention all the theorems and *include examples and visualizations in their talks*.

Lecture 6. *Toric varieties 1.* (11/4,)

Discuss affine toric varieties and convex rational polyhedral cones, following [KKMSD73, Chap.1 §1].

Lecture 7. *Toric varieties 2.* (11/11,)

Discuss toric varieties and fans (f.r.p.p. decompositions), following [KKMSD73, Chap.1 §2]. The proof of Sumihiro's theorem can be skipped.

Lecture 8. *Toric varieties 3.* (11/18,)

Discuss cohomology and convexity, following [KKMSD73, Chap.1 §3].

Lecture 9. *Toroidal embeddings.* (11/25,)

Define toroidal embeddings and explain the basic results based on the techniques of toric varieties, following [KKMSD73, Chap.2 §1,2].

Lecture 10. *Semistable reduction theorem in characteristic 0.* (12/2,)

Using Hironaka's resolution of singularities and the previous results, reduce the semistable reduction theorem in characteristic 0 to a construction of nice polyhedral subdivisions as in [KKMSD73, Chap.2 §3]. The construction is a combinatorial question, and Chapter 3 is devoted to the proof. Pick up several key arguments and guiding examples, and explain the main ideas and outline of the proof of the main theorem (Theorem 4.1) in the remaining time.

REFERENCES

- [Ber97] Pierre Berthelot, *Altérations de variétés algébriques (d'après A. J. de Jong)*, no. 241, 1997, Séminaire Bourbaki, Vol. 1995/96, pp. Exp. No. 815, 5, 273–311. MR 1472543
- [dJ96] A. J. de Jong, *Smoothness, semi-stability and alterations*, Inst. Hautes Études Sci. Publ. Math. (1996), no. 83, 51–93. MR 1423020
- [Ful93] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry. MR 1234037
- [Har03] Urs T. Hartl, *Semi-stable models for rigid-analytic spaces*, Manuscripta Math. **110** (2003), no. 3, 365–380. MR 1969007
- [KKMSD73] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics, vol. Vol. 339, Springer-Verlag, Berlin-New York, 1973. MR 335518