Cor of Stein factorization: $f: X \to Y$ proper (Noeth. over $X$ and $Y$)

$$\{\text{connected component of } f^{-1}(y)\} \xleftrightarrow{\cong} \{\text{max'l ideal of } (f_* \mathcal{O}_X)_y\}.$$

pf: by Stein factorization, LHS = pts of $(f_*\mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} \kappa(y)$.

So all we need: $A$ = local ring, $A \to B$ finite, then $\mathrm{Max}(B) = \mathrm{Spec}(B \otimes_A \kappa)$.

Cor. $f: X \to Y$ proper dominant (int'l Noeth. over $X$ and $Y$), so $k(Y) \subseteq k(X)$.

$\forall y \in Y$, $\#\{\text{conn'd comp of } f^{-1}(y)\} \leq \#\mathrm{Max}(\text{int'l closure of } \mathcal{O}_{Y,y} \text{ in } k(X))$.

Lemma: $f: X \to Y$, loc. f.type (loc. Noeth. over $X$ and $Y$). $X^{qf} := \{x \in X \mid \{x\} \text{ is clopen in } f^{-1}(f(x))\}$.

If $U \subseteq X$ open, then $X^{qf} \cap U = U^{qf}$.

pf: WMA: $Y = \mathrm{Spec}(k)$ is a field. ($f^{-1}(f(x)) \simeq X_{f(x)}$ as ~~top.~~ space)

Then $x \in X$ is loc. type$/k$, TFAE:

(1) $\{x\} \subseteq X$ clopen

(2) $\{x\} \subseteq X$ open & $\kappa(x)/k$ finite

if $x \in U \subseteq X$: $\{x\} \subseteq U$ clopen $\Updownarrow$ $\Leftrightarrow$ $\{x\} \subseteq U$ open & same

Prop. $f: X \to Y$ proper (Noeth. over $X$ and $Y$).

(1) if $f_* \mathcal{O}_X = \mathcal{O}_Y$ ($\Rightarrow f$ has conn'd fibers)

then $X^{qf} \subseteq X$ open & $f|_{X^{qf}}: X^{qf} \to Y$ is an open imm.

(2) $X^{qf} \subseteq X$ open in general.

pf: (1) $\Rightarrow$ (2) by Stein factorization.

Pf of (1): $X^{qf} = \{x \in X \mid \{x\} = f^{-1}(f(x))\}$, let $y = f(x)$.

$f$ proper $\Rightarrow \{f^{-1}(V) \mid y \in V \underset{\text{open}}{\subseteq} Y\}$ form a basis of open nbhds of $f^{-1}(y)$ $(= \{x\})$.

$$\Rightarrow \quad \mathcal{O}_{Y,y} \xrightarrow{\cong} (f_*\mathcal{O}_X)_y = \operatorname*{colim}_{y \in V} f_*\mathcal{O}_X(V) = \operatorname*{colim}_{y \in V} \mathcal{O}_X(f^{-1}(V)) = \mathcal{O}_{X,x}.$$

So $\exists$ affine nbhds

$$\begin{array}{ccc} \mathfrak{q} & & \\ \updownarrow & & \\ x \in \operatorname{Spec}(B) & \subseteq & X \\ & & \downarrow \\ \mathfrak{p} & & \\ \updownarrow & & \\ f(x) = y \in \operatorname{Spec}(A) & \subseteq & Y \end{array}$$

s.t.

① $A \xrightarrow{f^\#} B$ f. type of Noeth. rings.

② $S^{-1}A \otimes_A -$ applied to $f^\#$ becomes an isom,

where $S = A \setminus \mathfrak{p}$.

Exercise: show that $\exists\ g \in S$, s.t. $A[1/g] \xrightarrow{\cong} B[1/f^\#(g)]$.

Conclusion: we found affine nbhd

$$\begin{array}{ccc} x \in f^{-1}(V) & \subseteq & X \\ & & \downarrow \\ f(x) = y \in V & \subseteq & Y \\ \| & & \\ \operatorname{Spec}(A[1/g]) & & \end{array}$$

s.t. $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is an isom.

In particular ① $f^{-1}(V) \subseteq X^{qf}$.

② $f|_{X^{qf}} : X^{qf} \longrightarrow Y$ is injective + open map. & isom of $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_{X^{qf}}$ $\Big\}$ $\Rightarrow f|_{X^{qf}}$ is an open immn.

Defn : $X \longrightarrow Y$ is quasi-finite if f.type + fibers are finite.

Noeth. Noeth.

Thm (Chevalley): $f : X \longrightarrow Y$ proper + quasi-finite, then $f$ is finite.

Noeth. Noeth.

pf :
$$\begin{tikzcd} X \arrow[r, "\text{proper}", "f'"'] \arrow[dr, "f"'] & \underline{\mathrm{Spec}}(f_* \mathcal{O}_X) \arrow[d, "g", "\text{finite}"'] \\ & Y \end{tikzcd}$$

$X = X^{qf} \Rightarrow f'$ is an open immersion.

$\left.\begin{matrix} f \text{ proper} \\ g \text{ sep} \end{matrix}\right\} \Rightarrow f'$ proper.

Exercise: Show that $f'$ has dense image. (Hence $f'$ is an isom.).

~~Exercise~~ Cor : $f : X \longrightarrow Y$ birat'l, bijective, proper, $Y$ normal.

Noeth. int'l

Then $f$ is an isom.

Exercise/Thm: $f : X \longrightarrow Y$ f.type between loc. Noeth. schms.

$f$ is a closed immersion $\Longleftrightarrow$ $f$ is a proper monomorphism.

Thm (algebraic form of Zariski's Main Thm):

$A$ = Noeth. rng, $B$ = f.g. $A$-alg.

$\mathfrak{q} \subseteq B$ prime, $\mathfrak{p} :=$ corresp. prime in $A$.

- $\mathfrak{q}$ isol. in $A_{\mathfrak{p}} \otimes_A B$ $\Longleftrightarrow$ $\mathfrak{q}$ is both minimal & max'l among primes above $\mathfrak{p}$.

Then $\exists\, g \notin \mathfrak{q}$, a finite $A$-alg $C$, $\mathfrak{q}' \subseteq C$ above $\mathfrak{p}$, $h \notin \mathfrak{q}'$ s.t.

$B[1/g] \cong C[1/h]$ as $A$-alg.

pf: 
$$\begin{tikzcd} U := \mathrm{Spec}(B) \arrow[r, phantom, "\overset{\text{open}}{\subseteq}"] \arrow[dr] & X \arrow[r, hook] \arrow[d] \arrow[drr] & \mathbb{P}^n_A \\ & \mathrm{Spec}(A) & \mathbb{A}^n_A \arrow[u, phantom, "\subseteq"] & \mathrm{Spec}(C) \arrow[ll] \end{tikzcd}$$
Stein fact.

$q \in U^{qf} = X^{qf} \cap U$.

Exercise: finish the pf.

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Thm (Nagata compactification):
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, phantom, "\underset{\text{q-c. open}}{\cap}"] & S \\ \overline{X} \arrow[ur, "\text{proper}"'] & \end{tikzcd}$$
Assume: • $S$ qc qs.
• $f$ sep. + f.type.

Then can enlarge diagram.

Thm (Zariski's Main Thm): $f: X \longrightarrow Y$, sep. + f.type. (Noeth.)

Then $X^{qf} \subseteq X$ open, and
$$\begin{tikzcd} X^{qf} \arrow[r, phantom, "\overset{\text{open}}{\subseteq}"] \arrow[d, phantom, "\subseteq"] & X \arrow[r] & Y \\ Y' \arrow[urr, "\text{finite}"'] & & \end{tikzcd}$$

pf: Noeth. $\Rightarrow$ qc qs.
$$\begin{tikzcd} \overset{\text{intersection}}{=} X^{qf} \arrow[r, phantom, "\overset{\text{qc open}}{\subseteq}"] \arrow[d, phantom, "\cap"] & \overline{X}^{qf} \arrow[r, phantom, "\overset{\text{(qc.) open}}{\subseteq}"] \arrow[d, phantom, "\cap"] & Y' \\ X \arrow[r, phantom, "\subseteq"] & \overline{X} \arrow[r] \arrow[d] & Y' \arrow[dl] \\ & Y & \end{tikzcd}$$
$\square$

Cor: sep + q-f. $=$ finite $\circ$ open.

Exercise: find counterexample if we drop ~~sep~~.