$X \xrightarrow{f} Y$ l.f.t.

Defn. f is smooth at $x \in X$ if $(NL)_x \xrightarrow[\sim]{qis}$ proj mod. $[0]$.

(—— 〃 —— of rel dim r @ x, —— 〃 —— $r =$ rk of proj mod.)

$\iff \pi_1(NL_x) = 0$ & $(\Omega^1_{X/Y})_x$ is loc. free $/ \mathcal{O}_{X,x}$.

f is smooth if smooth at all $x \in X$. $\iff \pi_1(NL_f) = 0$ & $\Omega^1_{X/Y}$ loc. free.

Lemma: (1) smoothness is stable under composition.

(2) —— 〃 —— base change.

(3) smooth at $x \in X \Rightarrow \exists\, U =$ open nbhd of $x \in X$ s.t. $f|_U$ is smooth.

In particular, smooth locus is open in X.

pf: (1) NL sequence.

(2)
$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$
, recall $\Omega_{B'/A'} = \Omega_{B/A} \otimes_B B'$

$\pi_1(NL_{B'/A'}) \twoheadleftarrow \pi_1(NL_{B/A} \otimes^L_B B') = 0$ □.

(3).
$$\begin{tikzcd} I/I^2 \arrow[r, "Jac"] & \bigoplus B \cdot dx_i \arrow[r, two heads] & \Omega_{B/A} \arrow[l, dashed, bend right] \end{tikzcd}$$
I/I^2 f.g. $\Longrightarrow$ $\Omega_{B/A}$ f.pr. $\Rightarrow$ invert b, $\Omega_{B/A}[\frac{1}{b}]$ proj. ($\mathfrak{p} \not\ni b$)

So $I/I^2 \simeq \pi_1 \oplus \mathrm{Im}(Jac) \Rightarrow \pi_1$ f.g. $\Rightarrow$ invert $b' \notin \mathfrak{p}$, $\pi_1 = 0$. □

Defn. $A \to B$ ~~std~~ smooth if $A[\underline{x}] \twoheadrightarrow B$

$I = (f_1, \cdots, f_c)$ s.t. $\det\left(\frac{\partial f_i}{\partial x_j}\right)_{1 \le i,j \le c}$ is a unit in B.

Lemma: Std smooth is smooth & $I/I^2 \xleftarrow{\cong} \bigoplus_{i=1}^{c} B\cdot f_i$.

pf:

$$\begin{tikzcd} I/I^2 \arrow[r, "Jac"] & \bigoplus_{i=1}^{n} B\cdot dx_i \\ \bigoplus_{i=1}^{c} B\cdot f_i \arrow[u] \arrow[ur, hook, "{(GL_c(B),\ *)}"'] & \end{tikzcd}$$

□.

Prop. $A \longrightarrow B$ smooth at $\mathfrak{p} \subseteq B \Longrightarrow \exists\, b \notin \mathfrak{p}$ s.t. $A \to B[1/b]$ std sm.

pf: $I/I^2 \hookrightarrow \bigoplus B\cdot d\underline{x} \twoheadrightarrow \Omega_{B/A}$.

$\exists\, b \notin \mathfrak{p}$, $I/I^2[1/b]$ gen. by $\bar{f}_1, \dots, \bar{f}_c$ & (up to renumbering x_i's) $\det\left(\frac{\partial f_i}{\partial x_j}\right)_{c\times c}$ unit in $B[1/b]$.

$J = (f_1, \dots, f_c) \subseteq I \subseteq \tilde{\mathfrak{p}} \subseteq A[\underline{x}]$.

$(J + I^2)_{\tilde{\mathfrak{p}}} = I_{\tilde{\mathfrak{p}}} \Rightarrow (I/J)_{\tilde{\mathfrak{p}}} = I\cdot(I/J)_{\tilde{\mathfrak{p}}} \overset{中山}{\Longrightarrow} (I/J)_{\tilde{\mathfrak{p}}} = 0$

$\Longrightarrow I[1/P] = J[1/P]$ for some $P \notin \tilde{\mathfrak{p}}$.

Now: $\exists\, Q = P\cdot\tilde{b} \notin \tilde{\mathfrak{p}}$ s.t. $A \longrightarrow A[\underline{x}][y] \twoheadrightarrow A[\underline{x}][y] / \overbrace{(f_1, \dots, f_c, Q\cdot y - 1)}^{I'} \cong B[1/P\cdot b] = B[1/b'] =: B'$

& $\det\left(\frac{\partial f_i}{\partial x_j}\right)_{c\times c}$ unit in $\longrightarrow B'$.

$$\Longrightarrow \begin{tikzcd} I'/I'^2 \arrow[d, "Jac"'] & B'\cdot(Q\cdot y - 1) \oplus \bigoplus_{i=1}^{c} B'\cdot f_i \arrow[l, "\cong"'] \arrow[dl, "{(\bar{Q} = \text{unit},\ \cancel{GL_c}\ *)}"] \\ \bigoplus B'\cdot dy \oplus B'\cdot d\underline{x} & \end{tikzcd}$$

□.

smooth $\Longleftrightarrow$ Zar. open affine cover of X given by Std smooth.

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§ Deformation theory.

Goal: $X \xrightarrow{f} Y$ l.f.pr. Then f is smooth iff <u>formally smooth</u>

$\forall\ J \subseteq R \twoheadrightarrow R_0 := R/J$ w/ $J^2 = 0$.

$$\begin{tikzcd} \mathrm{Spec}(R_0) \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

Thm (Illusie)
$$\begin{tikzcd} V(\mathcal{I}) = S_0 \arrow[r, "g"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ S \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

$\mathcal{I}^2=0 \Rightarrow NL_{S_0/S} \simeq \mathcal{I}[1]$.

Functoriality $\Rightarrow Lg^*(NL_f) \xrightarrow{\alpha} NL_{S_0/S}$.

$$\{\text{possible } h\} \xrightarrow{\sim} \{\text{homotopy between } \alpha \ \& \ 0\}.$$

Idea: (1) Construct arrow: If h exists, then
$$\begin{tikzcd} S_0 \arrow[r, hook] \arrow[d, "i"'] & S \arrow[r, "h"] \arrow[d, equal] & X \arrow[d, "f"] \\ S \arrow[r, equal] & S \arrow[r] & Y \end{tikzcd} \Rightarrow \alpha \text{ factors through } NL_{S/S}=0.$$

(2) After constructing this arrow, we may work locally:
$$\begin{tikzcd} R_0 & B \arrow[l] & A[\underline{x}] \supseteq I \arrow[l, two heads] \arrow[dll, dashed, "\varphi"'] \\ R \arrow[u, two heads] & A \arrow[l] \arrow[u] \arrow[ur] & \\ J,\ J^2=0 \arrow[u, phantom, sloped, "\subseteq"] & & \end{tikzcd}$$

$$\begin{tikzcd} {[I/I^2 \otimes_B R_0} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{homotopy}"] & {\bigoplus R_0 \cdot d\underline{x}]} \arrow[d] \arrow[dl, dashed] \\ {[J} \arrow[r] & {0]} \end{tikzcd}$$
$\Updownarrow$
modification of φ s.t. "$\varphi + h$"$(I)=0$. □

Prop. $X \longrightarrow Y$ lfp., formally smooth $\Rightarrow$ smooth.

pf:
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B = A[\underline{x}]/I \arrow[d, "\cong"] \arrow[dl, dashed, "h"'] \\ A[\underline{x}]/I^2 \arrow[r] & B \end{tikzcd} \rightsquigarrow \begin{tikzcd} I/I^2 \arrow[r] \arrow[d, "\cong"] & \bigoplus B \cdot d\underline{x} \arrow[dl, dashed, "h"] \\ I/I^2 & \end{tikzcd}$$

formally smooth $\Rightarrow \exists h \Rightarrow \left[I/I^2 \xrightarrow{\text{Jac}} \bigoplus B \cdot d\underline{x} \right]$

$\Rightarrow NL \simeq$ direct summand of f.free $[0]$ □.

Prop. $X \longrightarrow Y$ l.f.p., smooth $\Rightarrow$ formally smooth.

pf:
$$\begin{tikzcd} S_0 = \mathrm{Spec}(R_0) \arrow[r, "g"] \arrow[d] & X \arrow[d, "f"] \\ S = \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

Illusie $\Rightarrow \exists$ extn class
$$0 \to J \to \mathcal{E} \to \Omega_{X/Y} \otimes_{\mathcal{O}_X} R_0 \to 0.$$

& choice of h is same as choice of splitting. $\square$

Prop. $X \xrightarrow{f} Y$ lfp. Then f smooth $\Longleftrightarrow$ flat + smooth fibers.

pf: $\Leftarrow$: Suffices to show $(NL_{B/A})_m \xrightarrow[\sim]{qis} \text{proj}\,[0]$ $\forall\, m \in \text{Max}(B)$.

$\mathfrak{p} := f(m)$.

$$\begin{array}{ccc} A & \longrightarrow & B \\ \cup\!\!\mathsf{I} & & \cup\!\!\mathsf{I} \\ \mathfrak{p} & & m \end{array}$$

[Lemma: (B, m) local

$K := [\text{ f.g. } B\text{-mod} \longrightarrow \text{free } B\text{-mod}]$

$K \xrightarrow[\sim]{qis} \text{proj}\,[0] \Longleftrightarrow \exists\, J \subseteq m$ $K \overset{L}{\underset{B}{\otimes}} B/J \xrightarrow[\sim]{qis} \text{proj } B/J\text{-mod}\,[0]$.

$$(NL_{B/A})_m \overset{L}{\underset{B_m}{\otimes}} B_m/\mathfrak{p}B_m \xrightarrow[\sim]{qis.} \left(NL_{(B\otimes_A \kappa(\mathfrak{p}))_{\bar{m}}/\kappa(\mathfrak{p})}\right). \quad \square$$

$$B_m/\mathfrak{p}B_m \simeq B_m \overset{L}{\underset{A_\mathfrak{p}}{\otimes}} \kappa(\mathfrak{p})$$

$\Rightarrow$: base change stable. Only need to show flat. smooth $\Rightarrow$ Zar. open cover by std sm.

$A \longrightarrow B = A[x_1, \cdots, x_n]/(f_1, \cdots, f_c)$. std smooth. $\Rightarrow$ flat.

WMA: (1) A (hence also B) Noeth. (by choosing f.t./$\mathbb{Z}$ models).

(2) A is local. (A, m, κ)

Step 1: $(\bar{f}_1, \cdots, \bar{f}_c)$ in $\kappa[x_1, \cdots, x_n]$ is reg. seq.

b/c $\underbrace{\kappa[x]/(\bar{f}_i)}$ has min. primes $\mathfrak{p}_i$, $\dim \Omega_{\kappa(\mathfrak{p}_i)/\kappa} = n-c$

$\Rightarrow$ eq. Krull dim $\geq n-c$. $\quad$ $\vee\!\mathsf{I}$ tr.d. $(\kappa(\mathfrak{p}_i)/\kappa)$

(always $\leq$)

$\Rightarrow$ eq. Krull dim $= n-c$ & reg. seq.

Step 2: Lemma: (A, m, κ) Noeth. local, $A \longrightarrow B$ flat, $f \in B$ Noeth.

If $\bar{f}$ in B/mB nonzerodiv., then

(1) f is a nonzerodiv.

(2) $A \longrightarrow B/f$ flat.

pf: (1) $\Rightarrow$ (2) by applying to $A/I \to B/IB \Rightarrow B/IB \xhookrightarrow{\cdot f} B/IB$

To see (1), WMA, B is local Noeth.

Now $0 \to m^n/m^{n+1} \longrightarrow A/m^{n+1} \longrightarrow A/m \to 0$ $\qquad - \otimes_A B$

$$0 \to m^n/m^{n+1} \otimes_{A/m} B/mB = m^nB/m^{n+1}B \longrightarrow B/m^{n+1}B \longrightarrow B/mB \to 0.$$

$\bar f$ inj $\qquad$ $\bar f$ inj $\qquad$ $\bar f$ inj.

$\Rightarrow \ker(f) \subseteq \bigcap_n m^nB \overset{\text{Krull}}{=\!=} 0.$ $\square$

Summary: $X \xrightarrow{f} Y$ lfp. TFAE.

(1) $NL_f \xrightarrow[\sim]{\text{qis}} \text{proj}[0]$

(2) Z. open cover by std smooth

(3) formally smooth

(4) flat + fiber smooth.

Exercise: • If lft. Then unramified $\Leftrightarrow$ formally unram.

• If l.f.pr. Then étale $\Leftrightarrow$ formally étale.

$\Leftrightarrow NL_f \xrightarrow[\sim]{\text{qis}} 0.$

$\Leftrightarrow$ flat + (fiber étale. $\Leftrightarrow$ fiber unram.)

• l.f.t./field smooth $\Leftrightarrow$ geometrically smooth

$\Leftrightarrow$ geometrically regular.